Theoretical aspects of relativistic geodesy

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Models of Gravity

Universität Bremen*

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Relativistic clock comparison effects effects

effect	term	on Earth	for satellites
longitudinal Doppler	v/c	negligible	$\lesssim 10^{-2}$ per day
transversal Doppler	v^2/c^2	Earth rotation	$\lesssim 10^{-5}$ s per day
Sagnac effect	$\omega\Omega\Sigma/c^2$	up to 10^{-13}	$\sim 10^{-7}$ s per orbit
1st order grav. redshift	$\Delta U/c^2$	up to 10^{-14}	$\sim 10^{-7}$ s per day
2nd order grav. redshift	$(\Delta U/c^2)^2$	negligible	$\sim 10^{-14}$ s per day
gravitational time delay	$\sim \frac{GM}{c^2} \ln \frac{r_1 r_2}{b^2}$	negligible	$\sim 10^{-11}$ s
gravitomagn. clock effect	J/Mc^2	measurable(?)	$\sim 10^{-7}$ s per orbit

relevant effects have to be included in TAI and in Galileo ($\sim$ 10 km/day)
in general, most of these effects cannot be strictly separated – this is possible only for weak fields

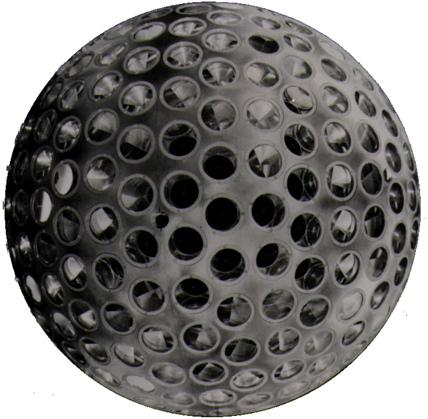
LAGEOS

LAGEOS (LAser GEOdynamics Satellite)

- Satellite equipped with laser reflectors
- high density – small air drag

science

- measuring Earth mass multipoles
- confirmation of Lense-Thirring effect with accuracy of approx. 10%
- improvement with LARES (Laser Relativity Satellite) in ca. 5 years with accuracy of approx. 1%



NASA

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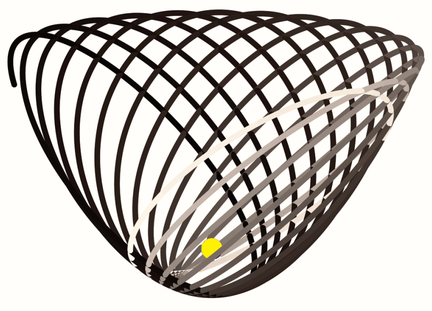
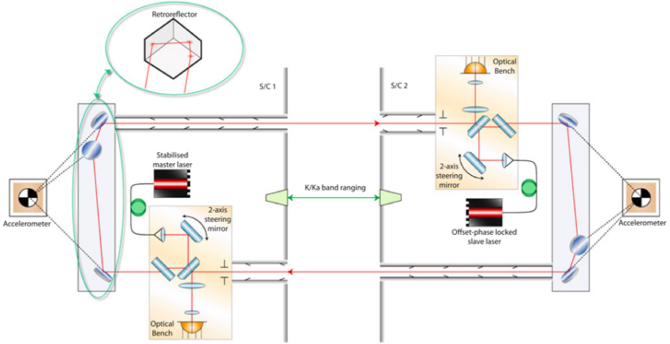


New method: Laser interferometry

GRACE-FO: laser interferometry (AEI Hannover)



precision ~ 10 nm
relativistic effects of the order $100\ \mu$m $\Rightarrow$ General Relativity has to be considered

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Theoretical framework

- **gravitational field** from matter: Einstein equation (Einstein, SBPAW 1915)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- **equation of motion** of a pointlike particle moving in the gravitational field: geodesic equation

$$0 = \frac{d^2 x^\mu}{ds^2} + \left\{ {}^{\mu}_{\rho\sigma} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}$$

  $\left\{ {}^{\mu}_{\rho\sigma} \right\}$ is the Christoffel symbol, and $ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$
  extended particles: Mathisson-Papapetrou-Dixon equations (Dixon 1968)

- **clock reading** = proper time

$$s = \int ds$$

  operationally defined through standard clocks (Perlick, GRG 1987), approximately (on Earth with highest precision) realized by atomic clocks

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Consequences

- effect on light rays
  - light deflection (VLBI, Gaia) Eddington
  - lensing Twin Quasar Q0957+561
  - shadows of black holes (EHT)
- orbital effects
  - perihelion shift (Mercury) Le Verrier
  - Lense-Thirring effect: spin-orbit coupling (LAGEOS) Ciufolini
  - back reaction effects (binary systems) Hulse-Taylor, grav. waves
- effects on extended bodies
  - Schiff effect: spin-spin coupling (GP-B) Everitt
- clock effects / effects on frequency
  - gravitational redshift Pound-Rebka, GP-A
  - gravitational time delay Cassini
- gravitational waves Abbott et al

+ all special relativistic effects: time dilation, Doppler effect, Sagnac effect, length contraction, aberration, ...

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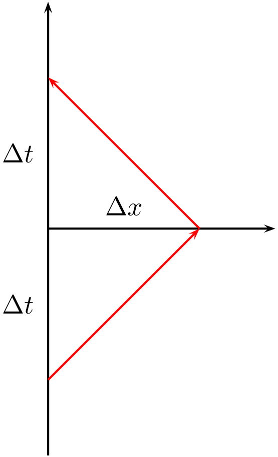
The standard clock

relative distance in rest space ($c = 1$)

$$\Delta x = \Delta t$$



The standard clock

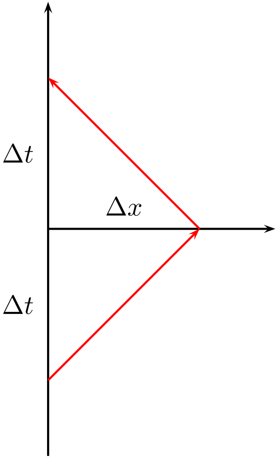
relative distance in rest space ($c = 1$)

$$\Delta x = \Delta t$$

relative velocity (invariant)

$$\dot{x} = \frac{dx}{dt} = \sqrt{\frac{-g(\Delta x, \Delta x)}{g(u,u)}}$$

relative acceleration $\ddot{x} = \ldots$ (complicated expression)



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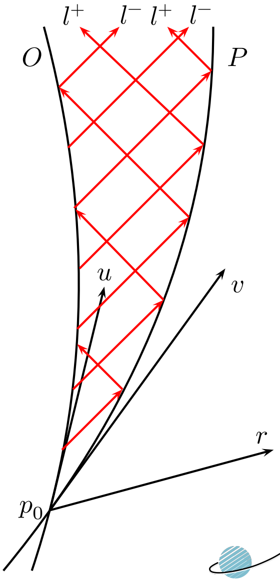

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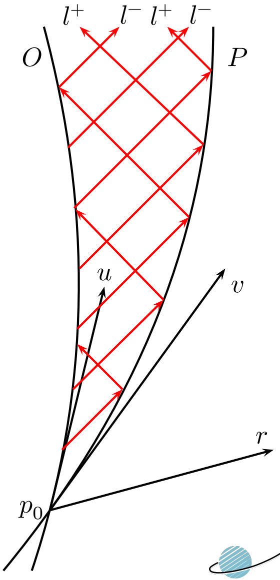

Definition (Perlick GRG 1987)

An observer (in arbitrary motion) is equipped with a standard clock if for all freely falling particles she measures the same

$$\frac{\ddot{x}}{1 - \dot{x}^2}$$

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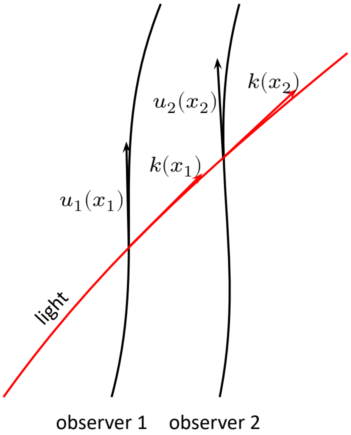
The gravitational redshift within GR



- light ray given by trajectory with tangent $l(\lambda)$ obeying $g(l,l) = 0$
- observer with standard clock given by trajectory with tangent $u(\tau)$ obeying $g(u,u) = 1$

measured frequency

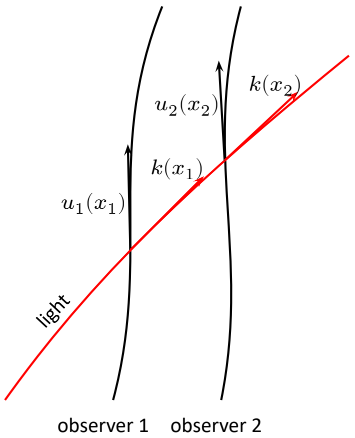
$$\nu = g_{\mu\nu} l^\mu u^\nu = k_\mu u^\mu \quad \text{with} \quad k_\mu = g_{\mu\nu} l^\nu$$

stationary gravitational field: ξ Killing vector $\Rightarrow$ $k(\xi) = const \Rightarrow$ gravitational redshift for stationary observer $u = e^{-\phi}\xi$

$$\frac{\nu_2}{\nu_1} = \frac{k(u_2)}{k(u_1)} = \frac{e^{\phi_2}}{e^{\phi_1}} = \sqrt{\frac{g_{tt}(r_2)}{g_{tt}(r_1)}} \approx 1 - \frac{GM}{c^2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

light needs not to propagate along geodesics – optical fibers are allowed

The gravitational redshift within GR



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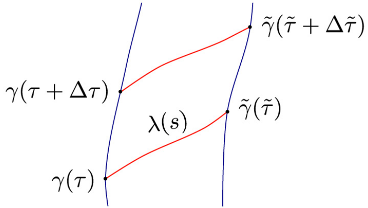
General relativistic definition of geoid I: Clocks

Basic notions

- k is the wave vector of a light ray
- u is the 4-velocity of an observer
- Measured frequency given by
  $\nu := k(u) = k_\mu u^\mu = g_{\mu\nu} k^\mu u^\nu$

The redshift defined by light ray can be related to a redshift potential ϕ



$$\phi(x) := \ln \frac{\tilde{\nu}(x)}{\nu(x_0)}, \qquad \nu = \frac{1}{\Delta\tau}$$

- possesses the correct post-Newtonian approximation
- Can be extended to light rays propagating in optical fibers (no geodesics) with known position dependent diffraction index

ϕ gives the redshift $\Rightarrow$

ϕ is a fully general relativistic geoid

Clock comparison with optical fibers

generally valid representation of metric

$$g = e^{2\phi}\left(-\left(cdt + \alpha_a(x)dx^a\right)^2 + \alpha_{ab}(x)dx^a dx^b\right)$$

light propagation through fiber ($\dot{(\cdot)} = \frac{d}{ds}(\cdot)$)

$$0 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \qquad \Leftrightarrow \qquad cdt + a_a dx^a = \sqrt{\alpha_{ab}dx^a dx^b}$$

gives coordinate travel time

$$\Delta t = t_2 - t_1 = \int_{t_1}^{t_2} dt = \frac{1}{c}\int\left(\sqrt{\alpha_{ab}\dot{x}^a\dot{x}^b} - \alpha_c\dot{x}^c\right)ds$$

this gives redshift (with $d\tau/dt = e^\phi$)

$$\frac{\nu}{\tilde{\nu}} = \frac{d\tilde{\tau}}{d\tau} = \frac{d\tilde{\tau}}{dt}\frac{dt}{d\tau} = \frac{e^{\tilde{\phi}}}{e^{\phi}} = \frac{\tilde{g}_{tt}}{g_{tt}}$$

Clock comparison with optical fibers

with refractive index we have a modified “metric”

$$g = e^{2\phi}\left(-\frac{1}{n^2(x)}\left(cdt + \alpha_a(x)dx^a\right)^2 + \alpha_{ab}(x)dx^a dx^b\right)$$

modified redshift

$$\frac{\nu}{\tilde{\nu}} = \frac{e^{\tilde{\phi}}}{e^{\phi}}\frac{n}{\tilde{n}}$$

with known refractive index we can determine the gravitational redshift potential

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Relativistic geoid II: Model of the Earth

Earth described by a continuum, relative velocity between constituents can be decomposed

$$v^\mu = \omega^\mu{}_\nu r^\nu + \sigma^\mu{}_\nu r^\nu + \frac{1}{3}\theta r^\mu$$

- a body is called rigid, if all spatial distances and angles between nearby particles remain constant

The rigid body

A non-expanding and shear-free congruence is called rigid.

the rigid body can still rotate and accelerate in a *time-dependent* way

Theorem

If for a rigid body the rotation is constant and the acceleration rotates with the rigid body, then

- the congruence is stationary,
- the acceleration can be derived from a potential.

Relativistic geoid II: Model of the Earth

Definition

A congruence is called stationary if u is proportional to a Killing vector field ξ with $\mathcal{L}_\xi g = 0$. This is also called an isometry.

- $\xi = e^\phi u$
- expansion $\theta = 0$, shear $\sigma = 0$
- $a_\mu = D_u u_\mu = -\partial_\mu \phi$
- potential is time independent: $D_u \phi = 0$

the acceleration of falling bodies (falling corner cubes, plumb lines) is given by a potential $\phi \Rightarrow$

ϕ is a fully general relativistic geoid

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The synthesis

Theorem

Both definitions of a geoid coincide

both, clocks and falling bodies, define the same geoid

measurements can be combined

The synthesis

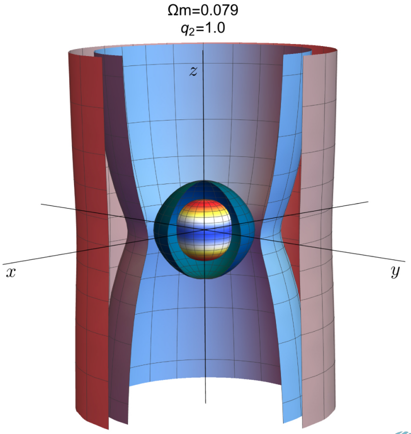
Theorem

Both definitions of a geoid coincide

both, clocks and falling bodies, define the same geoid
measurements can be combined

Philipp, Hackmann, Perlick, Puetzfeld, C.L., PRD 2017

Geoid in Erez-Rosen space-time



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Question

- how to measure the geoid from space, that is, with moving satellites $\rightarrow$
  - changing positions $\rightarrow$ changing gravitational redshift
  - varying velocity $\rightarrow$ changing Doppler shift: linear and quadratic

Clock comparison space – ground

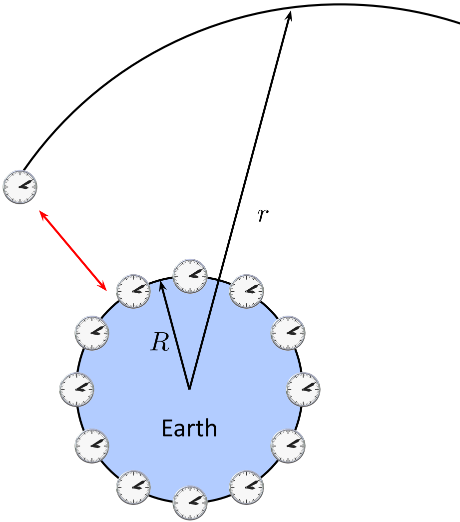
General problem: clock and frequency comparison between clock on ground (rotating) and clock on a satellite (orbiting the Earth)

- clock on rotating Earth
- satellite moving on geodesic
- electromagnetic signal between satellite and Earth moves on geodesic: emitter-receiver problem

simplified model

- radial signals
- Schwarzschild orbits

$$r(\varphi) = \frac{2M}{\wp(\varphi) + \frac{1}{6}}$$



Timing: special case

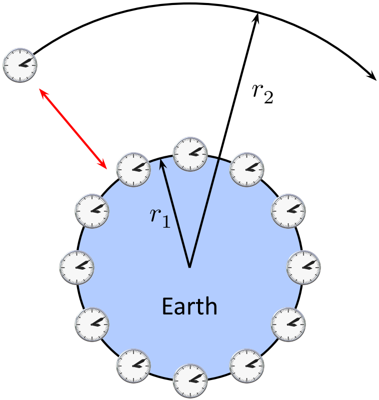
slightly simplified case

- clock orbiting the Earth in circular motion
- Schwarzschild geometry
- time comparison through vertical light rays

result

$$\frac{\nu_1}{\nu_2} = \frac{1}{\sqrt{1 + \frac{M}{r_2 - 3M}}} \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}}$$

has to be generalized to arbitrary links
for 2nd pN approximation of general case see Linet & Teyassandier, PRD 2002



Timing: special case II

- radially freely falling clock
- radial time comparison

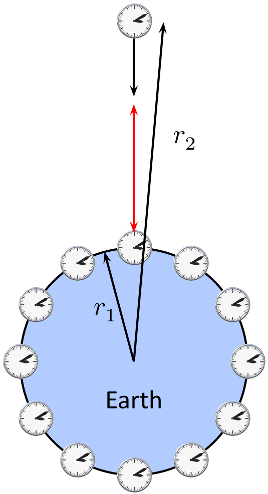
result

$$\frac{\nu_1}{\nu_2} = \frac{\sqrt{g_{00}(r_2)}}{\left(E - \sqrt{E^2 - g_{00}(r_2)}\right)} \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}}$$

E related to initial velocity of clock, where r_2 has to be determined as function of proper time from the geodesic equation

interpretation

- first factor: velocity of freely falling clock
- second factor: gravitational redshift



Timing: The general case

spherically symmetric space-time, stationary observer

$$\tilde{g}_{tt} = g_{tt}(\tilde{u}^t)^2 - \Omega^2 r_\oplus^2$$

general redshift

$$\frac{\nu_1}{\nu_2} = \left(u^t + \frac{u^r}{g_{tt}}\right) \sqrt{\frac{g_{tt}(r_\oplus)}{1 + r_\oplus^2 \Omega^2}}$$

with geodesic motion of u^μ

$$\frac{\nu_1}{\nu_2} = \sqrt{\frac{g_{tt}(r_\oplus)}{g_{tt}(r)}} \frac{E \pm \sqrt{E^2 - g_{tt}(r)\left(1 + \frac{L^2}{r^2}\right)}}{\sqrt{g_{tt}(r)}\sqrt{1 + r_\oplus^2 \Omega^2}}$$

where r is the solution of the geodesic equation: $r = r(\varphi)$, $r = r(t)$, $r = r(s)$

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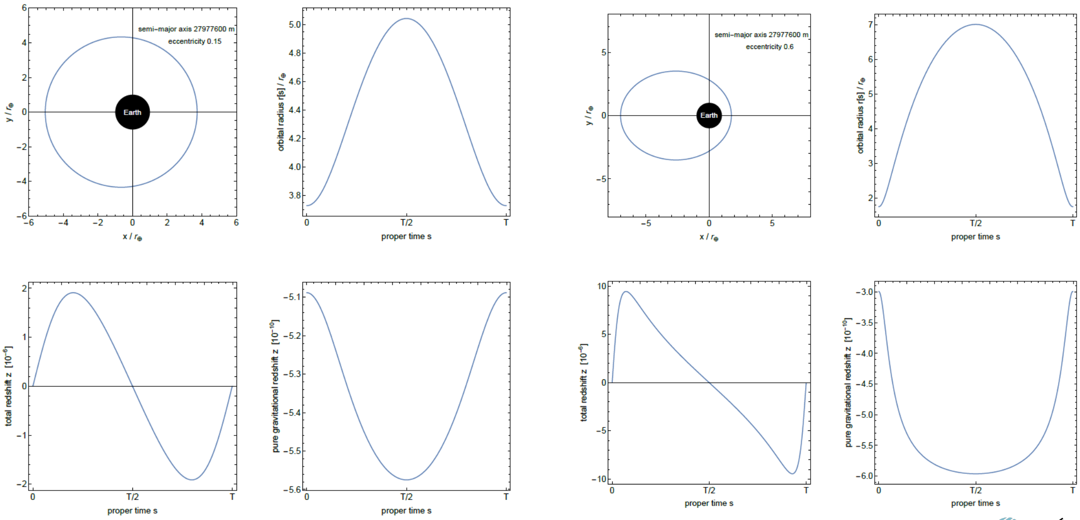
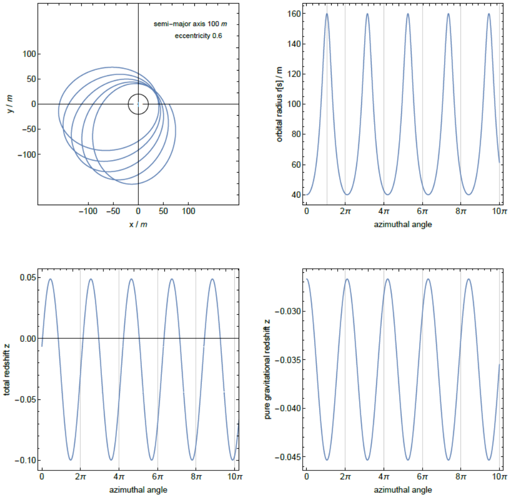
First analytic results



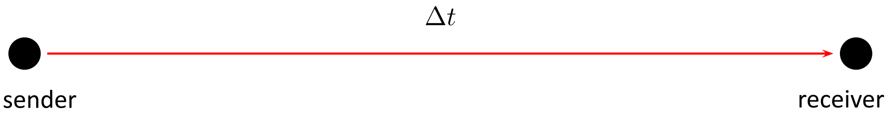
First analytic results



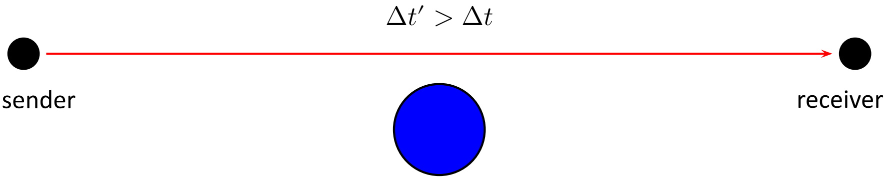
needs to be generalized to

- Kerr space-time
- space-time with multipole moments
- non-radial signal transmission

Gravitational time delay



Gravitational time delay



$$\Delta t = 2(1+\gamma) M \ln \frac{x_1 x_2}{b^2} \qquad \text{or} \qquad y = \frac{\nu(t)}{\nu} = 2(1+\gamma)\frac{M}{b}\frac{db}{dt}$$

b is impact parameter

- for the Sun $\Delta t \sim 10^{-4}$ s, $y \sim 10^{-9}$
- for the Earth $\Delta t \sim 10^{-11}$ s, $y \sim 10^{-12}$ – however: better statistics
- for positioning: corresponds to approx. 1 cm in distance

requires three or more frequencies in order to eliminate atmospheric influences

The gravito-magnetic clock effect

- rotating mass $\Rightarrow$ gravito-magnetic field

$$ds^2 = \ldots + \frac{2Mr}{r^2 + a^2\cos^2\vartheta} a \sin^2\vartheta \, d\varphi \, dt + \ldots$$

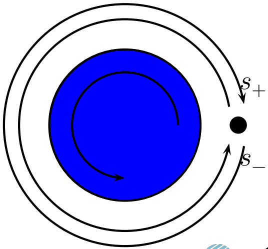
- from geodesic equation for circular orbits in equatorial plane

$$\frac{d\varphi}{dt} = \pm\Omega_0 + \Omega_{\text{Lense-Thirring}}$$

- difference of proper time for clocks on two counter-propagating satellites

$$s_+ - s_- = 4\pi \frac{J}{M} \sim 10^{-7} \ s$$

  - depends neither on G nor on r
  - becomes smaller for increasing inclination
  - vanishes for polar orbits
  - generalization for arbitrary orbits: Hackmann, C.L., PRD 2014



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Questions, challenges

- how to measure the geoid from space, that is, with non-stationary moving satellites
  one has to add Doppler shifts related to the satellite's velocity; precision limited by knowledge of velocity and position of the satellite
- what is the meaning of the other components, degree of freedom, of the gravitational field (as, e.g., the Kerr parameter)? Do these parameters define their own geoid?
- how to relate the geoid with the multipole moments of the gravitational field at infinity or with the mass multipoles of the Earth?

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Multipole moments

for a given mass distribution the Newtonian gravitational potential of this gravitating body is given by

$$U(\vec{r}) = G \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3V'$$

asymptotic expansion (expansion for large distances) of the integral

$$U(\vec{r}) = \frac{M}{r} + \frac{\vec{p} \cdot \vec{r}}{r^3} + \frac{Q_{ij} r_i r_j}{r^5} + \frac{Q_{ijk} r_i r_j r_k}{r^7} + \ldots$$

with

$$M = G \int \rho(\vec{r}) d^3V \qquad \text{monopole}$$

$$\vec{p} = G \int \vec{x} \rho(\vec{x}) d^3V \qquad \text{dipole}$$

$$Q_{ij} = G \int (3 r_i r_j - r^2 c \delta_{ij}) \rho(\vec{r}) d^3V \qquad \text{quadrupole}$$

$\Rightarrow$ moments defined through asymptotic expansion = multipole moments of mass distribution

General relativistic multipoles

Newton gravity

$$U = \frac{GM}{r}\left(1 - \sum_{l=1}^{\infty}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^{l}\left(C_{lm}\cos(m\varphi) + S_{lm}\sin(m\varphi)\right)P_{lm}(\cos\vartheta)\right)$$

coefficients describing the relativistic gravitational field

- static axially symmetric space-time: multipole expansion of gravity well defined, Weyl space-times
- multipoles in stationary axially symmetry (Quevedo & Mashhoon, PRD 1991)

relativistic mass multipoles of the gravitating mass

- multipoles defined at spatial infinity for stationary axially symmetric space-times (Geroch, Hanson, 1970)
- not defined for non-axially symmetric space-times
- mass multipoles as integrals over the energy momentum tensor (Dixon, 1964 - 1974), requires a frame of reference

due to non-linearity of the Einstein theory, no unique relation known

can be resolved in post-Newton approximation (Thorne 1980, Damour, Soffel, Xu, 1992 - 1994)

Weyl space-times

- Weyl space-times given by
$$ds^2 = e^{2\psi} dt^2 - e^{-2\psi}\left(\rho^2 d\varphi^2 - e^{2\gamma}(d\rho^2 + dz^2)\right)$$
where ψ obeys $\Delta\psi = 0$ with solutions
$$\psi = \sum_{l=0}^{\infty} \frac{c_l}{r^{l+1}} P_l(\cos\vartheta)$$
- possesses two Killing vectors $\xi = \partial_t$ and $\eta = \partial_t + \Omega\partial_\varphi$
- coordinate transformation $\rho^2 = m^2(x^2-1)(1-y^2)$, so that
$$\psi = \sum_{l=0}^{\infty} (-1)^{l+1} q_l Q_l(x) P_l(y)$$
with q_l mass multipole moments (can be read off from Newton limit) and potentials
$$e^{2\phi_{\text{stat}}} = e^{2\psi}\,, \qquad e^{2\phi_{\text{rot}}} = e^{2\psi} - m^2\Omega^2 e^{-2\psi}(x^2-1)(1-y^2)$$

Weyl space-times

- representation of Weyl metric in terms of multipole moments

$$ds^2 = e^{2\psi} dt^2 - m^2 e^{-2\psi} \left((x^2-1)(1-y^2) d\varphi^2 + e^{2\gamma}(x^2-y^2) \left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2} \right) \right)$$

- geoid given by

$$W_0 = \sqrt{e^{2\psi} - m^2 \Omega^2 e^{-2\psi}(x^2-1)(1-y^2)} - 1$$

with

$$2\psi = \sum_{l=0}^{\infty} (-1)^{l+1} q_l P_l(y) \left(P_l(x) \ln \frac{x+1}{x-1} - 2 \sum_{k=0}^{[\frac{l}{2}-\frac{1}{2}]} \frac{2l-4k-1}{(l-k)(2k+1)} P_{l-2k-1}(x) \right)$$

question: non-axially symmetric space-times, mass distributions?

Weyl space-times

- representation of Weyl metric in terms of multipole moments

$$ds^2 = e^{2\psi}dt^2 - m^2 e^{-2\psi}\left((x^2-1)(1-y^2)d\varphi^2 + e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right)\right)$$

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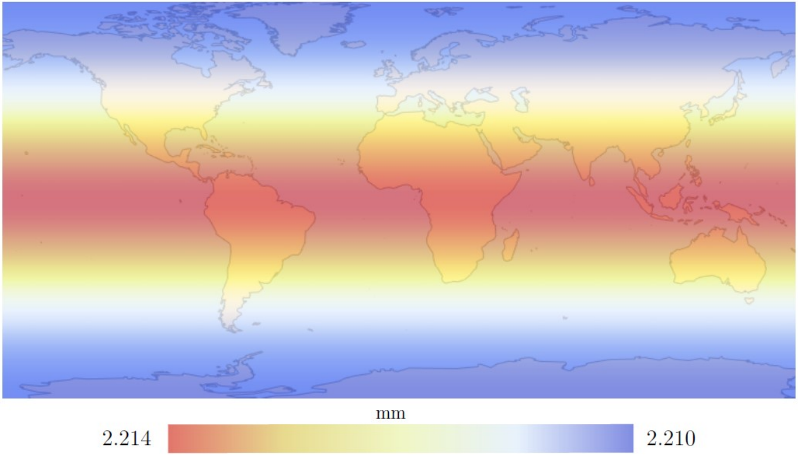
question: non-axially symmetric space-times, mass distributions?

Relativistic geoid

comparison of location of pN and Newtonian geoid, for axially symmetric quadrupole moment



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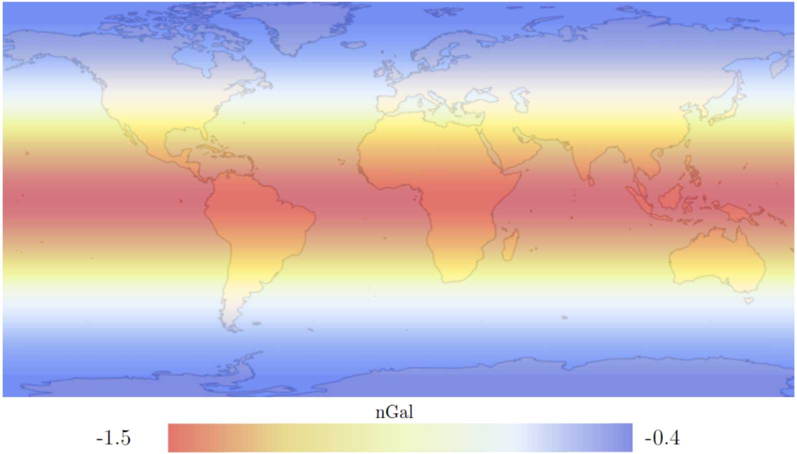
Relativistic geoid

comparison of acceleration for Schwarzschild and Kerr



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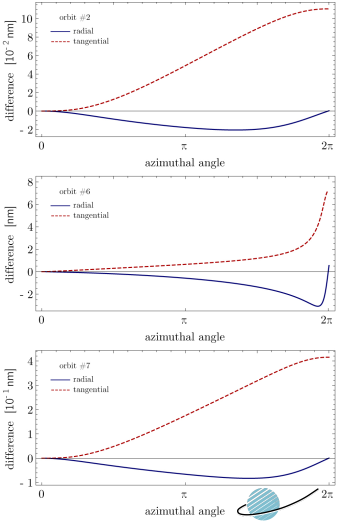
General relativistic orbit modelling

- general relativistic treatment of orbits is needed
- only for very few space-times analytic solutions of the geodesic equations are known (Hackmann, Kagramanova, Kunz, C.L., EPL 2009)
- for realistic space-times one needs numerical codes
- successful implementation of post-Newton effects of Schwarzschild gemetry in XHPS developed at ZARM and DLR-RY (Philipp, Wöske, Biskupek, Hackmann, mai, List, Rievers, C.L., submitted)
- also comparison with analytical Lie series approach
- implementation for clocks under way
- challenge: implementation for general space-times

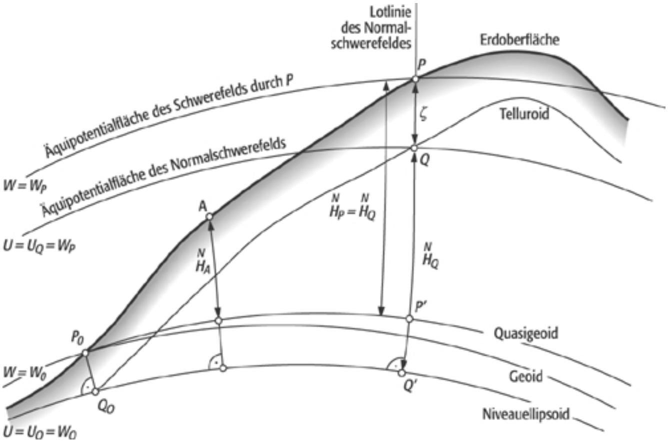


ZARM

Contents

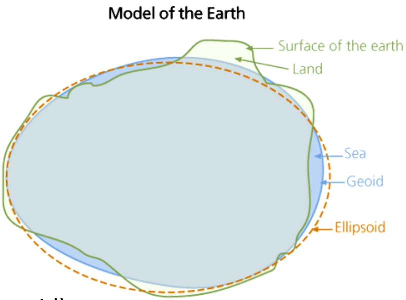
The relativistic quasi-geoid

- problem: determine the geoid inside mountains
- requires to solve gravitational field equation
- is fine for Newton, is complicated for Einstein field equation (nonlinear system - is under consideration)



Contents

The ellipsoid



the non-relativistic ellipsoid - various definitions

- ellipsoid with best fit to the surface of the Earth (geometry)
- ellipsoid with best fit to the geoid of the Earth (gravity)
- shape of a figure of equilibrium of an ideal fluid (self gravity)

general relativistic generalizations?

- best fit to general relativistic geoid is feasible (in terms of minimizing the invariant volume between the ellipsoid and the geoid)
- rotating figure of equilibrium - no analytical solution known
- is there an influence of the other components of the gravitational field (like, e.g., the Kerr parameter) on the definition of the ellipsoid?
- do we need further ellipsoids for the other gravitational degrees of freedom?

do we need an ellipsoid?

- ellipsoid is used by GPS - necessary?

Contents

Summary and outlook

Summary

- new measurement schemes require general relativistic treatment of geodesy
- general relativistic definition of the geoid

Outlook - challenges

- determination of the geoid using satellites
- where are the other gravitational degrees of freedom?
- general relativistic multipoles of the gravitational field vs. general relativistic mass multipole moments
- measurements with satellites

Thank you for your attention

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