Relativistic Corrections for Time and Frequency Transfer in Optical Fibres

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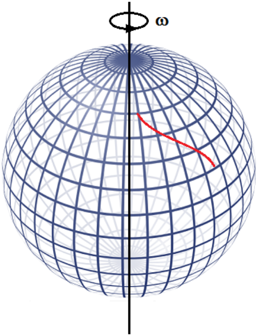
Project background and motivation

- EMRP project International Timescales with Optical Clocks (ITOC)
- Comparisons of distant optical clocks
- Theoretical description of TT and FT with accuracy required by optical clocks

Ojectives

- Fully relativistic description of signal propagation in optical fibres
- Deriving formulas for signal propagation time in optical fibre with 1 ps accuracy
- Deriving formulas for frequency change during signal propagation in optical fibre with relative accuracy of 10^{-18}



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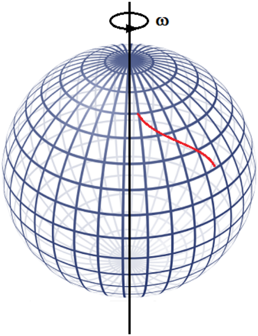

Outline

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Fibre time transfer [Gersl et al., 2015]

$$T_{AB} = \frac{1}{c}\int_0^L n \, \mathrm{d}l + \frac{1}{c^2}\int_0^L \mathbf{v}\cdot\mathbf{s}_l \, \mathrm{d}l + \frac{1}{c^3}\int_0^L n\left(W + v^2/2\right)\mathrm{d}l + O(c^{-4})$$

where n is the fibre effective refractive index, $\mathbf{v}$ is the velocity vector field of the fibre in the GCRS frame, $\mathbf{s}_l$ is the tangent vector field of the fibre, $v^2 = \mathbf{v}\cdot\mathbf{v}$, and all quantities are evaluated at $t = t_B$.

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Fibre time transfer: the Sagnac term

- for two-way time transfer only the Sagnac term remains:

$$\frac{1}{2}(T_{34} - T_{12}) = \frac{1}{c^2}\int_0^L \mathbf{v}\cdot\mathbf{s}_l \, \mathrm{d}l$$

- Let's introduce a Terrestrial frame which rotates rigidly with respect to the spatial part of GCRS with angular velocity vector $\boldsymbol{\omega}$ such that:

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{x} + \mathbf{v}_R$$

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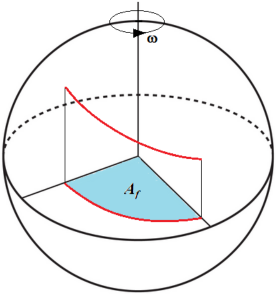

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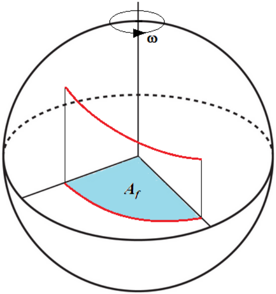

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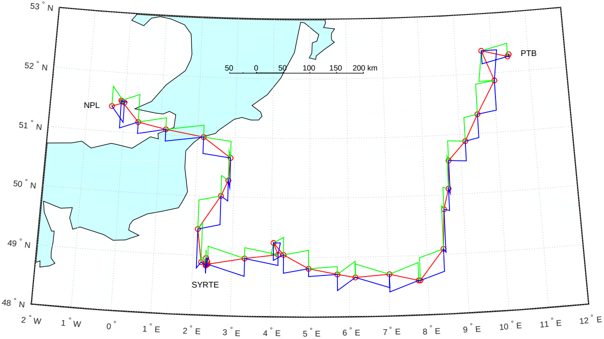
Sagnac delays in the REFIMEVE+ network



fibre link	Length/km	Correction/ps
PTB-SYRTE	1401	3976 ± 27
NPL-SYRTE	813	1214 ± 6

Parameters uncertainty for fibre time transfer

Parameter	Uncertainty
Fibre length (1-way only)	0.2 mm
Refractive index (1-way only)	3×10^{-10}
Fibre endpoints position	200 m
Fibre inner points position	600 m
Fibre velocity in co-rotating frame	9 cm/s
Earth angular velocity	~ 0.01 % (relative)
Gravitational plus centrifugal potential (1-way only)	~ 30 % (relative)

Table : Input parameters and their maximal uncertainties sufficient for 1 ps uncertainty in time transfer. The values were obtained for situations where the sensitivity of a correction to a parameter is maximized and they are calculated for 1000 km long fibre [Gersl et al., 2015].

Outline

Clock frequency comparison

- Determination of the frequency ratio f_A/f_B between the proper frequencies f_A and f_B delivered by clocks A and B
- In practice this is achieved using a transmission of photons from A to B and the formula
$$\frac{f_A}{f_B} = \frac{f_A}{\nu_A}\frac{\nu_A}{\nu_B}\frac{\nu_B}{f_B},$$
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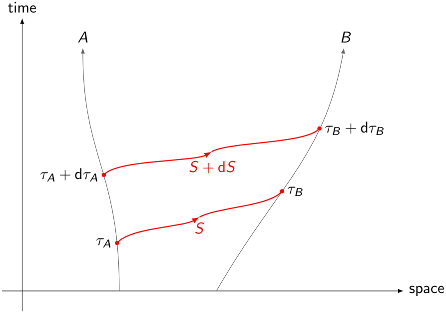
Fundamental relations of frequency transfer



- The proper frequency measured by $M = \{A, B\}$ is:

$$\nu_M = \frac{1}{2\pi} \frac{\mathrm{d}S}{\mathrm{d}\tau_M} \tag{1}$$

- Introducing the covariant wave vector $k_\alpha^M = (\partial_\alpha S)_M$ and the four-velocity $u_M^\alpha = \mathrm{d}x_M^\alpha / \mathrm{d}\tau_M$:

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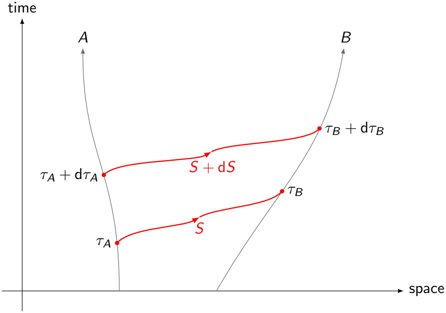

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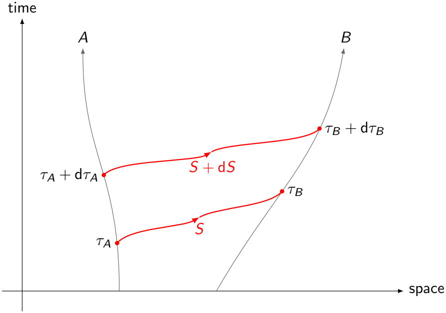

- Introducing $v^i = dx^i/dt$ and $\hat{k}_i = k_i/k_0$, it is usually written as:

$$\frac{\nu_A}{\nu_B} = \frac{u_A^0}{u_B^0}\frac{k_0^A}{k_0^B}\frac{1+\frac{\hat{k}_i^A v_A^i}{c}}{1+\frac{\hat{k}_i^B v_B^i}{c}}$$

- from (1) we deduce:

$$\frac{\nu_A}{\nu_B} = \frac{d\tau_B}{d\tau_A}$$
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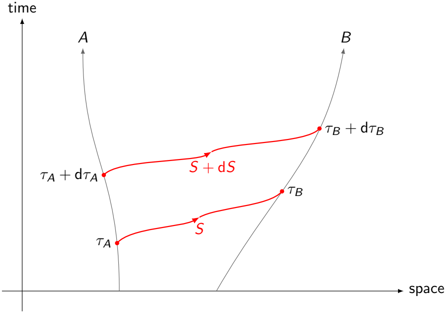

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Splitting of the terms

- One term depends depends only on the state (velocity and location) of the receiver and emitter

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The proper time / coordinate time relation in GCRS

- The scalar potential $W(X^\alpha) \equiv W(cTCG, \mathbf{X})$ of the GCRS metric can be split into three parts: contribution of the Earth itself, of tidal forces, and of inertial forces [Soffel et al., 2003]

$$W = W_{\mathrm{E}} + W_{\mathrm{tidal}} + W_{\mathrm{inertial}}$$

- Then the relation between proper time and TCG is [Wolf and Petit, 1995]:

$$\frac{\mathrm{d}\tau}{\mathrm{d}TCG} = 1 - \frac{1}{c^2}\left[W_{\mathrm{E}}(X^\alpha) + \frac{v^2}{c^2}\right. \\ \left. + \bar{W}(x^\alpha_{\mathrm{E}} + X^\alpha) - \bar{W}(x^\alpha_{\mathrm{E}}) - \bar{W}_{,k}(x^\alpha_{\mathrm{E}})X^k\right]$$

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$$\frac{\mathrm{d}t_B}{\mathrm{d}t_A} = 1 + \frac{1}{c}\int_0^L \left(\frac{\partial n}{\partial t} + n\alpha\frac{\partial T}{\partial t}\right)\mathrm{d}l + \frac{1}{c^2}\int_0^L \frac{\partial \mathbf{v}\cdot\mathbf{s}_l}{\partial t}\mathrm{d}l$$

where T is the temperature and other quantities are defined in the section on time transfer [Gersl et al., 2015]

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Fibre propagation

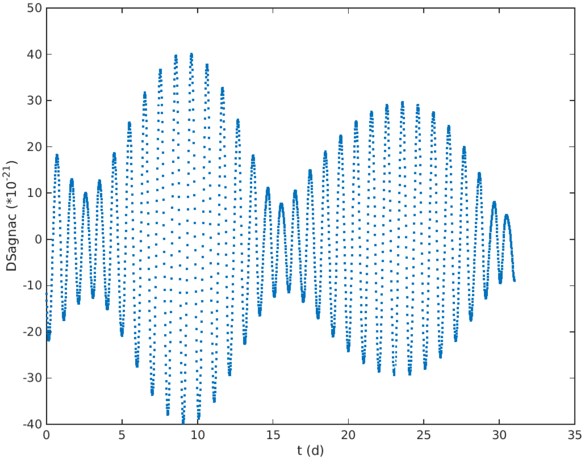
Magnitudes of the different terms

Effect	Correction
Variations in length and refractive index due to temperature changes (1-way only)	$\sim 10^{-13}$
Coriolis acceleration of the fibre due to velocity of the Earth tides	8×10^{-20}
Euler acceleration of the fibre due to angular acceleration of the Earth rotation	4×10^{-20}
Acceleration of the Earth tides	3×10^{-20}

Table : The corrections depend on processes in the whole fibre and they are calculated per 1000 km of the fibre length [Gersl et al., 2015]

Fibre propagation

The PTB-SYRTE link: Earth rotation signal



It corresponds to a variation of around 5 fs over 12 hours in the time transfer

Fibre propagation

Parameters uncertainty

Parameter	Uncertainty
Time derivative of the fibre temperature (change of length and refractive index; 1-way only)	3×10^{-11} K/s
Fibre velocity in co-rotating frame	0.6 mm/s
Fibre acceleration in co-rotating frame	9×10^{-8} m/s^2
Fibre position	$>$ Earth radius
Earth angular velocity	$>$ 100 % (relative)
Earth angular acceleration	$>$ 100 % (relative)

Table : Input parameters and their maximal uncertainties sufficient for 10^{-18} relative uncertainty in frequency transfer. The values were obtained for situations where the sensitivity of a correction to a parameter is maximized and they are calculated for 1000 km long fibre [Gersl et al., 2015].

Literature I

Gersl, J., Delva, P., and Wolf, P. (2015).
Relativistic corrections for time and frequency transfer in optical fibres.
Metrologia 552

Soffel, M., Klioner, S. A., Petit, G., Wolf, P., Kopeikin, S. M., Bretagnon, P., Brumberg, V. A., Capitaine, N., Damour, T., Fukushima, T., Guinot, B., Huang, T.-Y., Lindegren, L., Ma, C., Nordtvedt, K., Ries, J. C., Seidelmann, P. K., Vokrouhlický, D., Will, C. M., and Xu, C. (2003).
The IAU 2000 resolutions for astrometry, celestial mechanics, and metrology in the relativistic framework: Explanatory supplement.
The Astronomical Journal, 126(6):2687.

Wolf, P. and Petit, G. (1995).
Relativistic theory for clock syntonization and the realization of geocentric coordinate times.
Astronomy and Astrophysics, 304:653.