First Meeting of IAG JWG2.1

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Theoretical problems of relativistic geodesy

- Reference ellipsoid
- Normal gravity field
- Relativistic equations of motion
- Relativistic GNSS
- Relativistic IERS
- Relativistic TAI
- Relativistic time transfer and clock synchronization
- Beyond 1 Post-Newtonian approximation
- Very advanced topics - gravitational wave astronomy with clocks connected by optical fiber links, local measurement of the Hubble constant with clocks, etc.

Post-Newtonian Metric

We impose the harmonic gauge condition on the metric tensor

$$\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}g^{\alpha\beta}\right)=0\,. \tag{1}$$

The stationary post-Newtonian metric is

$$g_{00} = -1+\frac{2V}{c^2}+\frac{2}{c^4}\left(\Phi-V^2\right)+O\left(c^{-6}\right)\,, \tag{2}$$

$$g_{0i} = -\frac{4V^i}{c^3}+O\left(c^{-5}\right)\,, \tag{3}$$

$$g_{ij} = \delta_{ij}\left(1+\frac{2V}{c^2}\right)+O\left(c^{-4}\right)\,, \tag{4}$$

where the gravitational potentials satisfy the Poisson equation

$$\Delta V = -4\pi G\epsilon\,, \tag{5}$$

$$\Delta V^i = -4\pi G\epsilon v^i\,, \tag{6}$$

$$\Delta\Phi = -4\pi G\left(2\epsilon v^2+2\epsilon V+3p\right)\,, \tag{7}$$

with ϵ, p, and v^i being the energy density, pressure and velocity of matter. The energy density ϵ relates to mass density by

$$\epsilon=\rho^*\left(1+\frac{\Pi}{c^2}\right)\,, \tag{8}$$

where Π is the internal energy of matter per unit mass and ρ^* is the mass density of baryons per a unit of coordinate volume element. It is related in the post-Newtonian approximation to the local mass density ρ by equation

$$\rho^* = \rho + \frac{\rho}{c^2}\left(\frac{1}{2}v^2 + 3V\right) . \tag{9}$$

The internal energy Π is related to pressure p and the local density ρ by thermodynamic equation

$$d\Pi + pd\left(\frac{1}{\rho}\right) = 0 , \tag{10}$$

and equation of state, $p = p(\rho)$.

In the stationary spacetime, the mass density ρ^* obeys the *exact* equation of continuity

$$\frac{\partial\left(\rho^* v^i\right)}{\partial x^i} = 0 . \tag{11}$$

Velocity of rigidly rotating fluid is

$$v^i = \varepsilon^{ijk}\Omega^j x^k , \tag{12}$$

where Ω^i is the constant angular velocity. Replacing velocity v^i in (11) with (12), and differentiating, reveals that

$$v^i \frac{\partial \rho^*}{\partial x^i} = 0 , \tag{13}$$

which means that velocity of the fluid is tangent to the surfaces of constant density ρ^*. Equation (13) is also consistent with the case of the constant density ρ^* in a fixed coordinate system. We shall make use of this assumption in the rest of the paper.

Post-Newtonian reference ellipsoid (interior solution)

Level surface of uniformly rotating perfect fluid of homogeneous density ρ takes on the form of the *Maclaurin* biaxial ellipsoid

$$\frac{\sigma^2}{a^2} + \frac{z^2}{b^2} = 1 \qquad \text{where } \sigma^2 \equiv x^2 + y^2$$

PN theory tells us that the shape of such a uniformly rotating fluid body is to be a spheroid

$$\frac{\sigma^2}{a^2} + \frac{z^2}{b^2} = 1 + \kappa\left[K_1 \frac{\sigma^2}{a^2} + K_2 \frac{z^2}{b^2} + B_1 \frac{\sigma^4}{a^4} + B_2 \frac{z^4}{b^4} + B_3 \frac{\sigma^2}{a^2}\frac{z^2}{b^2}\right]$$

where $\kappa = \frac{\pi G \rho a^2}{c^2} \simeq 5.2\times10^{-10}$. This equation can be recast to the following form

$$\frac{\sigma^2}{{r_e}^2} + \frac{z^2}{{r_p}^2} = 1 + \kappa(B_1 + B_2 - B_3)\frac{\sigma^2}{a^2}\frac{z^2}{b^2}$$

where $r_e = a\left[1 + \frac{1}{2}\kappa(K_1 + B_1)\right]$ - equatorial radius, and $r_p = b\left[1 + \frac{1}{2}\kappa(K_2 + B_2)\right]$ - polar radius.

The residual gauge freedom is described by a PN coordinate transfromation

$$x'^{\alpha} = x^{\alpha} + \kappa\xi^{\alpha}(x)$$

where the gauge functions ξ^{α} obey the Laplace equation

$$\Delta\xi^{\alpha} = 0$$

Solution of this equation are harmonic polynomials. In the case of PN ellipsoid they are

$$\xi^1 = hx + p\frac{x}{a^2}\left(\sigma^2 - 4z^2\right) ,$$

$$\xi^2 = hy + p\frac{y}{a^2}\left(\sigma^2 - 4z^2\right) ,$$

$$\xi^3 = kz + q\frac{z}{b^2}\left(3\sigma^2 - 2z^2\right) ,$$

Transformation of the PN ellipsoid equation to the new coordinates is form-invariant. It transforms the coefficients in the equation of the PN ellipsoid

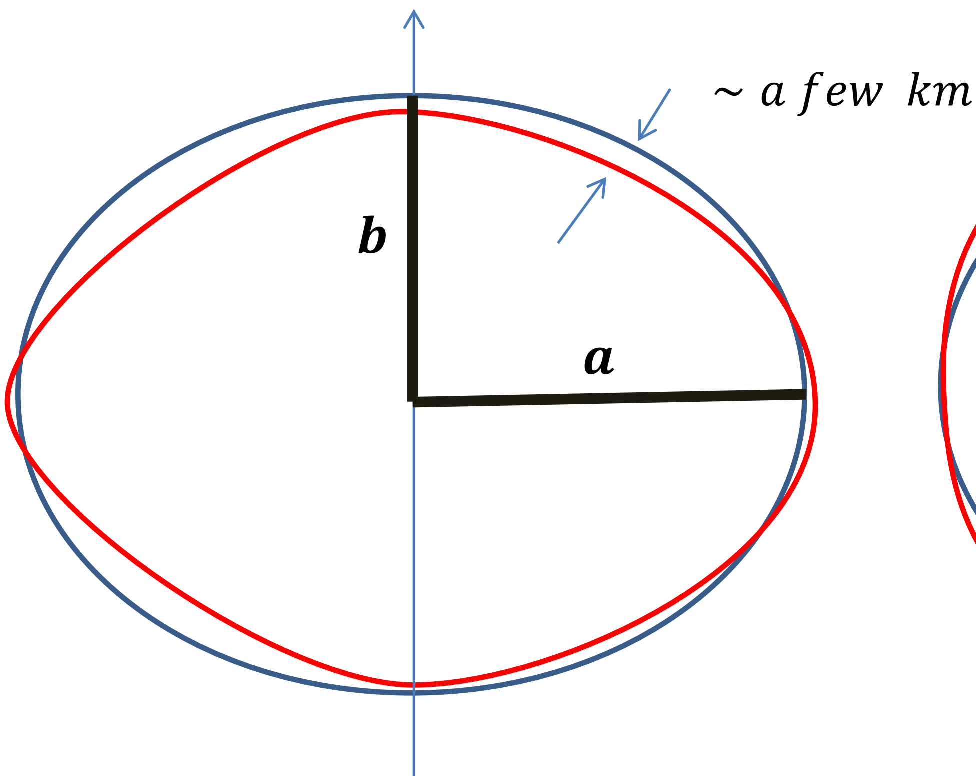
$$K_1 \to K_1 + 2h \qquad K_2 \to K_2 + 2k$$

$$B_1 \to B_1 + 2p \qquad B_2 \to B_2 - 4h$$

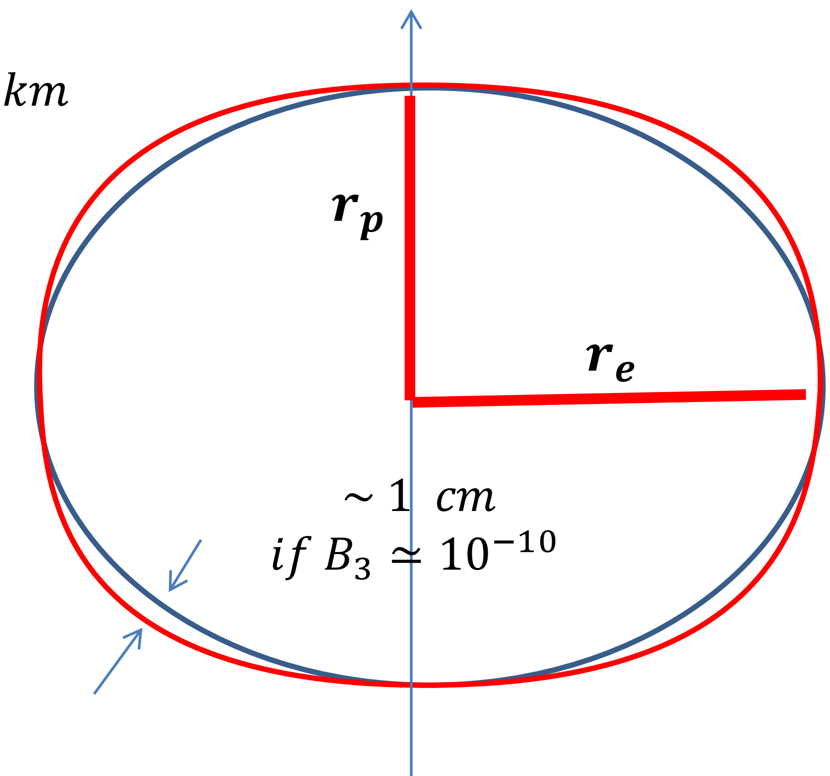
$$B_3 \to B_3 - 8p\frac{b^2}{a^2} + 6q\frac{a^2}{b^2}$$

Gauge freedom and the choice of coordinates

Arbitrary K_1, K_2, B_1, B_2, B_3



$K_1 = K_2 = B_1 = B_2 = 0, \qquad B_3 \neq 0$



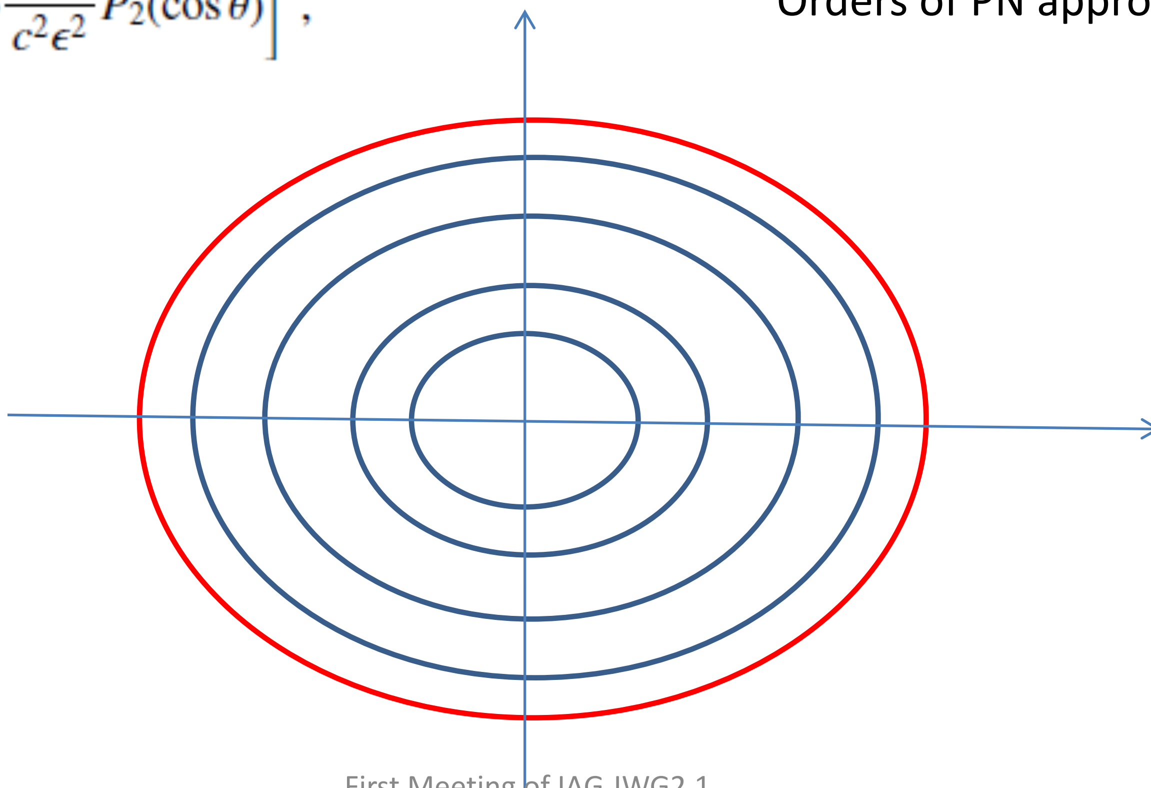
The problem has four degrees of freedom corresponding to the three infinitesimal coordinate transformations having four free parameters. Hence, we can arbitrary fix four out of the five constants in the mathematical equation of the PN ellipsoid. The only non-vanishing parameter can be chosen to be $B_3 \equiv B$.

Non-homogeneous fluid with ellipsoidal distribution of density

$$\rho = \rho_0 \left[1 + \frac{1}{c^2} \mathcal{A} \pi G \rho_c a^2 \frac{q_1}{\varkappa^2} \left(\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} \right) \quad \right]$$

$$\sigma_s = \frac{1}{\varkappa} \left[1 + \mathcal{B} \frac{\omega^2 a^2}{c^2 \epsilon^2} P_2(\cos\theta) \right] ,$$

Allows to keep the level surface as a biaxial ellipsoid at least, in 1 PN approximation (perhaps in all Orders of PN approximations



The normal gravity field
(exterior solution)

Coordinates

Ellipsoidal

$$x = \alpha\sqrt{1+\sigma^2}\sin\theta\cos\phi\ ,$$
$$y = \alpha\sqrt{1+\sigma^2}\sin\theta\sin\phi\ ,$$
$$z = \alpha\sigma\cos\theta\ ,$$

Spherical

$$x = R\sin\Theta\cos\Phi\ ,$$
$$y = R\sin\Theta\sin\Phi\ ,$$
$$z = R\cos\Theta\ .$$

Coordinate transformation

$$\sigma = \sqrt{\frac{r^2-1}{2}\left(1+\sqrt{1+\frac{4r^2\cos^2\Theta}{(r^2-1)^2}}\right)}, \qquad \cos\theta = \frac{r\cos\Theta}{\sqrt{\frac{r^2-1}{2}\left(1+\sqrt{1+\frac{4r^2\cos^2\Theta}{(r^2-1)^2}}\right)}}\ .$$

Green's Function in Elliptical Coordinates

$$G(x,x') = \frac{1}{|x-x'|} = \begin{cases} \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=-n}^{m=n}\frac{(n-|m|)!}{(n+|m|)!} q_{n|m|}(\sigma')\, p_{n|m|}(\sigma)\, Y^*_{nm}(\hat{x}')Y_{nm}(\hat{x})\,, & (\sigma \le \sigma')\,, \\ \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=-n}^{m=n}\frac{(n-|m|)!}{(n+|m|)!} p_{n|m|}(\sigma')\, q_{n|m|}(\sigma)\, Y^*_{nm}(\hat{x}')Y_{nm}(\hat{x})\,, & (\sigma' \le \sigma)\,, \end{cases}$$

Legendre polynomials from imaginary argument

Legendre functions from imaginary argument

Spherical harmonics

Scalar gravitational potential

$$V = V_N[\rho_0, \mathcal{S}] + V_N[\delta\rho_0, \mathcal{S}] + V_N[\rho_0, \delta\mathcal{S}] + \frac{1}{c^2} V_{pN}$$

$$V = \mathcal{E}_0 \left[q_0(\sigma) + q_2(\sigma) P_2(\cos\theta)\right] + \frac{1}{c^2} \mathcal{E}_4 \left[q_2(\sigma) P_2(\cos\theta) + q_4(\sigma) P_4(\cos\theta)\right]$$

$$\mathcal{E}_0 \equiv G m_N \left\{1 + \frac{1}{10c^2}\left[(\mathcal{A} - 10)\frac{9q_1(\varkappa^{-1})}{2\varkappa} G m_N + (\mathcal{B} + 4)\omega^2 a^2\right]\right\} ,$$

$$\mathcal{E}_4 \equiv \frac{G m_N}{10}\left\{(10 - \mathcal{A})\frac{9q_1(\varkappa^{-1})}{7\varkappa} G m_N + \frac{9\mathcal{B}}{28}\left(3 + \frac{30}{\varkappa^2} + \frac{35}{\varkappa^4}\right)\omega^2 a^2 - \frac{6}{7}\left(3 + \frac{5}{\varkappa^2}\right)\omega^2 a^2\right\} ,$$

The second eccentricity

Scalar multipole moments

$$V = \mathcal{E}_0 \left[q_0(\sigma) + q_2(\sigma) P_2(\cos\theta) \right] + \frac{1}{c^2} \mathcal{E}_4 \left[q_2(\sigma) P_2(\cos\theta) + q_4(\sigma) P_4(\cos\theta) \right]$$

$$q_0(\sigma) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{P_{2n}(\cos\vartheta)}{r^{2n+1}} ,$$

$$q_2(\sigma) P_2(\cos\theta) = - \sum_{n=1}^{\infty} \frac{2n(-1)^n}{(2n+1)(2n+3)} \frac{P_{2n}(\cos\vartheta)}{r^{2n+1}} ,$$

$$q_4(\sigma) P_4(\cos\theta) = \sum_{n=2}^{\infty} \frac{4n(n-1)(-1)^n}{(2n+1)(2n+3)(2n+5)} \frac{P_{2n}(\cos\vartheta)}{r^{2n+1}} .$$

$$V = \frac{GM}{R} \left[1 - \sum_{n=1}^{\infty} J_{2n} \left(\frac{a}{R} \right)^{2n} P_{2n}(\cos\vartheta) \right] ,$$

$$GM = \mathcal{E}_0 ,$$

$$J_{2n} = \frac{3(-1)^{n+1} \epsilon^{2n}}{(2n+1)(2n+3)} \left[1 - \frac{14}{3c^2} \frac{n}{2n+5} \frac{\mathcal{E}_4}{\mathcal{E}_0} \right] , \qquad (n \geq 1)$$

ϵ – the first eccentricity

$$\epsilon = \frac{\sqrt{a^2 - b^2}}{a^2}$$

Vector gravitational potential

$$V^i = G\int \frac{\rho(x')v^i(x')}{|x-x'|}d^3x' \ .$$

Each element of a rigidly rotating fluid has velocity, $v^i(x) = \varepsilon^{ijk}\omega^j x^k$, where ε^{ijk} is the Levi-Civita symbol, so that (114) can be written as follows

$$V^i = \varepsilon^{ijk}s^j\mathcal{D}^k \ , \tag{116}$$

where s^i is the unit vector along the angular velocity vector, $s^i = \omega^i/\omega$, the Cartesian vector $\mathcal{D}^k = \{\mathcal{D}^x, \mathcal{D}^y, \mathcal{D}^z\}$ is given by

$$\mathcal{D}^k = G\omega\int \frac{\rho(x')x'^k d^3x'}{|x-x'|} \ . \tag{117}$$

We denote, $\mathcal{D}^+ \equiv \mathcal{D}^x + i\mathcal{D}^y$. In the ellipsoidal coordinates (117) one has

$$\mathcal{D}^+ = G\omega\rho_0\alpha\int \frac{\sqrt{1+\sigma'^2}\sin\theta' e^{i\phi'}d^3x'}{|x-x'|} \ . \tag{118}$$

We had chosen the Cartesian coordinates such that the angular velocity, $\omega^i = (0,0,\omega)$, so that vector potential $V^i = (V^x, V^y, V^z)$ has $V^z = 0$ [2]. The remaining two components of the vector potential can be combined together

$$V^+ = V^x + iV^y = i\mathcal{D}^+ \ . \tag{119}$$

Equation (119) reveals that calculation of the vector potential is reduced to calculation of the integral (118) which depends on the point of integration, and is separated in to the internal and external solutions which we discuss below.

Integration with respect to the radial coordinate yields

$$\mathcal{D}^+ = \mathcal{D}e^{i\phi} \ ,$$

where

$$\mathcal{D} = -2\pi G\omega\rho_0\alpha^3\frac{(1+\varkappa^2)^2}{5\varkappa^5}\left[q_{11}(\sigma)P_{11}(\cos\theta) + \frac{1}{6}q_{31}(\sigma)P_{31}(\cos\theta)\right] \ .$$

Vector multipole moments

$$\mathcal{D} = -\frac{3}{10} G M \omega \frac{(1+\varkappa^2)}{\varkappa^2} \left[q_{11}(\sigma) P_{11}(\cos\theta) + \frac{1}{6} q_{31}(\sigma) P_{31}(\cos\theta) \right]$$

The products of the spheroidal harmonics entering (150) are expanded into spherical harmonics as follows:

$$q_{11}(\sigma) P_{11}(\cos\theta) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} \frac{P_{2n+1,1}(\cos\vartheta)}{r^{2n+2}} ,$$

$$q_{31}(\sigma) P_{31}(\cos\theta) = -8 \sum_{n=1}^{\infty} \frac{3n(-1)^n}{(2n+1)(2n+3)(2n+5)} \frac{P_{2n+1,1}(\cos\vartheta)}{r^{2n+2}} .$$

Replacing these expansions to (150) and reducing similar terms, we obtain

$$\mathcal{D} = -\frac{3GMa^2\omega}{R^2} \sum_{n=0}^{\infty} \frac{(-1)^n \epsilon^{2n}}{(2n+1)(2n+3)(2n+5)} \left(\frac{a}{R}\right)^{2n} P_{2n+1,1}(\cos\vartheta) .$$

$$\mathcal{D} = -\frac{GS}{2R^2} \sum_{n=0}^{\infty} S_{2n+1} \left(\frac{a}{R}\right)^{2n} P_{2n+1,1}(\cos\vartheta) ,$$

$$S = \frac{2}{5} M a^2 \omega ,$$

$$S_1 = 1 ,$$

$$S_{2n+1} = \frac{15(-1)^n \epsilon^{2n}}{(2n+1)(2n+3)(2n+5)} , \qquad (n \geq 1)$$

$$V^i = \frac{G}{2} \frac{(S \times x)^i}{R^3} \left[1 + 15 \sum_{n=1}^{\infty} \frac{(-1)^n \epsilon^{2n}}{(2n+1)(2n+3)(2n+5)} \left(\frac{a}{R}\right)^{2n} \frac{dP_{2n+1}(\cos\vartheta)}{d\cos\vartheta} \right] ,$$

The normal gravity field (i)

$$d\tau^2 = -\left(g_{00} + \frac{2}{c} g_{0i} v^i + \frac{1}{c^2} g_{ij} v^i v^j\right) dt^2 \ ,$$

$$\frac{d\tau}{dt} = 1 - \frac{W}{c^2} + O\left(c^{-6}\right)$$

$$W = \frac{1}{2} v^2 + V + \frac{1}{c^2}\left(\frac{1}{8} v^4 + \frac{3}{2} v^2 V - 4 v^i V^i - \frac{1}{2} V^2\right)$$

$$W(\sigma, \theta) = \mathcal{W}_0(\sigma) + \mathcal{W}_2(\sigma) P_2(\cos\theta) + \frac{1}{c^2} \mathcal{W}_4(\sigma) P_4(\cos\theta) \ ,$$

The normal gravity field (ii)

$$\mathcal{W}_0(\sigma) = \frac{1}{3}\omega^2\alpha^2(1+\sigma^2) + Gmq_0(\sigma)$$
$$+\frac{1}{c^2}\omega^2\alpha^2(1+\sigma^2)\left\{\frac{1}{15}\omega^2\alpha^2(1+\sigma^2) + Gm\left[q_0(\sigma) - \frac{1}{5}q_2(\sigma)\right]\right\}$$
$$-\frac{Gm}{c^2}\left\{\frac{8}{15}\omega^2 a^2\left[q_0(\sigma) + q_2(\sigma)\right] + \frac{1}{2}Gm\left[q_0^2(\sigma) + \frac{1}{5}q_2^2(\sigma)\right]\right\} ,$$

$$\mathcal{W}_2(\sigma) = -\frac{1}{3}\omega^2\alpha^2(1+\sigma^2) + Gmq_2(\sigma) +$$
$$+\frac{1}{c^2}\left\{\mathcal{E}_4 q_2(\sigma) - \omega^2\alpha^2(1+\sigma^2)\left[\frac{2}{21}\omega^2\alpha^2(1+\sigma^2) + Gm\left(q_0(\sigma) - \frac{5}{7}q_2(\sigma)\right)\right]\right\}$$
$$+\frac{Gm}{c^2}\left\{\frac{8}{15}\omega^2 a^2\left[q_0(\sigma) - \frac{5}{49}q_2(\sigma) - \frac{54}{49}q_4(\sigma)\right] - Gmq_2(\sigma)\left[q_0(\sigma) + \frac{1}{7}q_2(\sigma)\right]\right\} ,$$

$$\mathcal{W}_4(\sigma) = \mathcal{E}_4 q_4(\sigma) + \frac{1}{35}\omega^2\alpha^2(1+\sigma^2)\left[\omega^2\alpha^2(1+\sigma^2) - 18Gmq_2(\sigma)\right]$$
$$-\frac{9}{35}Gm\left\{Gmq_2^2(\sigma) - \frac{16}{7}\omega^2 a^2\left[q_2(\sigma) + q_4(\sigma)\right]\right\} ,$$

The Post-Newtonian Somigliana Formula

$$\gamma_n = \left(1 + \frac{1}{2c^2}\omega^2 a^2 \sin^2\theta\right)\left(-\frac{1}{\alpha}\sqrt{\frac{1+\sigma^2}{\sigma^2+\cos^2\theta}}\frac{\partial W}{\partial\sigma}\right)_{\sigma=1/\varkappa} ,$$

$$\gamma_n = \left(1 + \frac{1}{2c^2}\omega^2 a^2 \sin^2\theta\right)\sqrt{\frac{1+\sigma^2}{\sigma^2+\cos^2\theta}}\left[\gamma_0 + \gamma_2 P_2(\cos\theta) + \frac{1}{c^2}\gamma_4 P_4(\cos\theta)\right] ,$$

$$\gamma_0 \equiv -\frac{1}{\alpha}\frac{\partial W_0}{\partial\sigma}\bigg|_{\sigma=1/\varkappa} , \qquad \gamma_2 \equiv -\frac{1}{\alpha}\frac{\partial W_2}{\partial\sigma}\bigg|_{\sigma=1/\varkappa} , \qquad \gamma_4 \equiv -\frac{1}{\alpha}\frac{\partial W_4}{\partial\sigma}\bigg|_{\sigma=1/\varkappa} .$$

$$\gamma_n = \frac{a\gamma_a \sin^2\theta + b\gamma_b \cos^2\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} - \frac{1}{8c^2}\frac{4\omega^2 a^2 (a\gamma_a - b\gamma_b) + 35 b\gamma_4}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\sin^2\theta\cos^2\theta .$$

The Kerr Metric and Multipoles

$$ds^2 = \left[-1 + \frac{2V_K}{c^2} - \frac{2V_K^2}{c^4} + \frac{1}{c^4}\frac{2G^2 m_K^2 \cos^2\theta}{(\varsigma^2 + \cos^2\theta)^2}\right] c^2 dt^2 - \frac{8V_K^i}{c^2} dt dx^i + \left(1 + \frac{2V_K}{c^2}\right)\delta_{ij} dx^i dx^j \,,$$

$$V_K = \frac{G m_K \varsigma}{\varsigma^2 + \cos^2\theta}\,, \qquad V_K^i = \frac{c V_K}{2\alpha_K}\frac{(s \times x)^i}{1 + \varsigma^2}\,,$$

$$V_K = \frac{GM}{R}\left[1 - \sum_{n=1}^{\infty} J_{2n}^K \left(\frac{a}{R}\right)^{2n} P_{2n}(\cos\vartheta)\right]$$

$$J_{2n}^K = (-1)^{n+1}\epsilon_K^{2n} = (-1)^{n+1}\left(\frac{\omega a}{c}\right)^{2n}\,, \qquad (n \geq 1)$$

$$V_K^i = \frac{G}{2}\frac{(S \times x)^i}{R^3}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n \epsilon_K^{2n}}{2n+1}\left(\frac{a}{R}\right)^{2n}\frac{dP_{2n+1}(\cos\vartheta)}{d\cos\vartheta}\right]\,,$$

THANK YOU !