General relativistic geodesy

Towards an exact formalism

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APPLIED SPACE TECHNOLOGY
AND MICROGRAVITY
ZARM

Outline

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Clocks and clock comparison

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General relativistic geodesy
- The general relativistic geoid
- Open questions

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General Relativity

- equation for the **gravitational field**: Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- **equation of motion** of a pointlike particle moving in the gravitational field: geodesic equation

$$0 = \frac{d^2 x^\mu}{ds^2} + \left\{ {}^{\mu}_{\rho\sigma} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}$$

  $\left\{ {}^{\mu}_{\rho\sigma} \right\}$ is the Christoffel symbol, and $ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$
  extended particles: Mathisson-Papapetrou-Dixon equations

- **clock reading** = proper time, defined by geometry only

$$s = \int ds$$

  operationally defined through standard clocks (Perlick, GRG 1987), approximately realized by atomic clocks

Distinguished clocks

It is possible to build distinguished clocks

A geometric standard clock

- light clock = optical resonator
- Perlick's standard clock

result

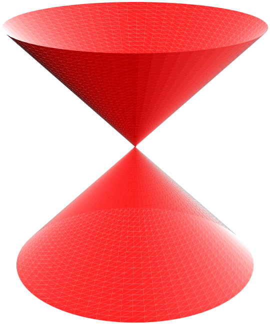
- reading of clock $ds^2 = \int g_{\mu\nu} dx^\mu dx^\nu$ along path of clock

- Enables transport of time unit
- Atomic clocks are standard clocks (if curvature is not too large, Parker & Pimentel, PRD 1982)

Prerequisite: Validity of Special and General Relativity

The physical basis

- The velocity of light does not depend on the velocity of the source
- does also not depend on frequency, polarization
- there is only one light ray emitted from one one point
  $\Rightarrow$ there is only one (!) velocity of light
- light is a property, a geometrical structure of space-time, does not depend on properties of light

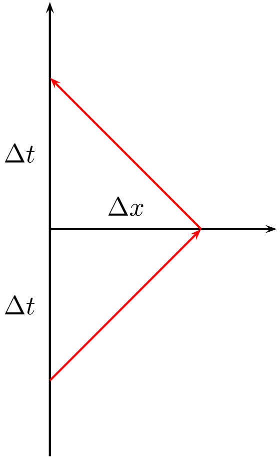


The underlying basic principles (principles of space-time physics) of geodesy and the precision of instruments require tests of the foundations of geodesy (Weak Equivalence Principle, Universality of Gravitational Redshift, Local Lorentz Invariance, ...)
E.g. better clocks always have to confirm gravitational redshift, etc. with the better precision

The standard clock

relative distance in rest space ($c = 1$)

$$\Delta x = \Delta t$$



The standard clock

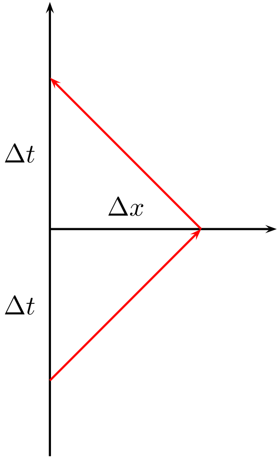
relative distance in rest space ($c = 1$)

$$\Delta x = \Delta t$$

relative velocity (invariant)

$$\dot{x} = \frac{dx}{dt} = \sqrt{\frac{-g(\Delta x, \Delta x)}{g(u, u)}}$$

relative acceleration $\ddot{x} = \ldots$
(complicated)



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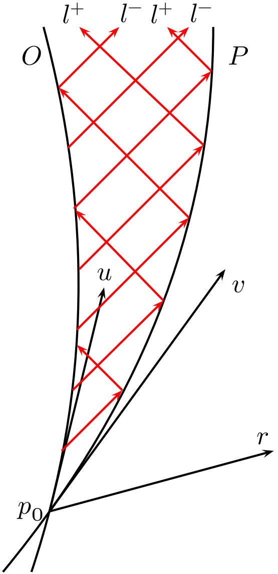

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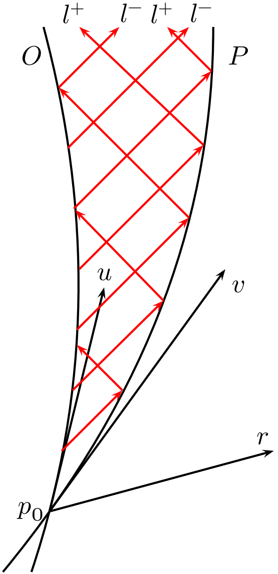
Definition (Perlick, GRG 1987)

An observer equipped with a standard clock measures for all freely falling particles the same $\dfrac{\ddot{x}}{1 - \dot{x}^2}$



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Relativistic clock comparison effects effects

effect	term	on Earth	for satellites
longitudinal Doppler	v/c	negligible	$\leq 10^{-2}$ per day
transversal Doppler	v^2/c^2	Earth rotation	$\leq 10^{-5}$ s per day
Sagnac effect	$\omega\Omega\Sigma/c^2$	up to 10^{-13}	$\sim 10^{-7}$ s per orbit
1^{st} order grav. redshift	$\Delta U/c^2$	up to 10^{-14}	$\sim 10^{-7}$ s per day
2^{nd} order grav. redshift	$(\Delta U/c^2)^2$	negligible	$\sim 10^{-14}$ s per day
gravitational time delay	$\sim \frac{GM}{c^2} \ln \frac{r_1 r_2}{b^2}$	negligible	$\sim 10^{-11}$ s
gravitomagn. clock effect	J/Mc^2	measurable(?)	$\sim 10^{-7}$ s per orbit

relevant effects have to be included in TAI and in GNSS

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Experiments

- first order gravitational redshift
  Pound & Rebka, PRL 1960: confirmation $\sim 1\%$
  Hafele & Keating, Nature 1968: confirmation $\sim 10\%$
  Vessot, Levine et al, GRG 1978, PRL 1980: GP-A, confirmation $\sim 10^{-4}$
  ZARM and SYRTE are working on Galileo data
- gravitational time delay
  Berttotti et al, Nature 2005; confirmation $\sim 10^{-5}$
- proposal: second order redshift for clocks in space
  Teyssandier & Linet, PRD 2002
- proposal: gravitomagnetic clock effect (beyond Schwarzschild and post-Newton)
  Hackmann, CL, PRD 2015
  could me measurable in space for GNSS satellites ($\sim 10^{-7}$ s per orbit)
  too small for measurements on ground

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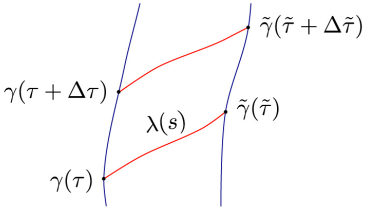
General relativistic definition of geoid I: Clocks



Basic notions

- k is the wave vector of a light ray
- u is the 4-velocity of an observer
- Measured frequency given by
  $\nu := k(u) = g_{\mu\nu} k^\mu u^\nu$

The redshift defined by light ray can be related to a redshift potential ϕ

$$\phi(x) := \ln \frac{\tilde{\nu}(x)}{\nu(x_0)}\,, \qquad \nu = \frac{1}{\Delta\tau}$$

- possesses the correct post-Newtonian approximation
- Can be extended to light rays propagating in optical fibers (no geodesics) with known position dependent diffraction index

ϕ gives the redshift $\Rightarrow$

ϕ is a fully general relativistic geoid

Clock comparison with optical fibers

metric

$$g = e^{2\phi}\left(-\left(cdt + \alpha_a(x)dx^a\right)^2 + \alpha_{ab}(x)dx^a dx^b\right)$$

light propagation through fiber ($(\dot{\cdot}) = \frac{d}{ds}(\cdot)$)

$$0 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \qquad \Leftrightarrow \qquad cdt + a_a dx^a = \sqrt{\alpha_{ab}dx^a dx^b}$$

gives coordinate travel time

$$\Delta t = t_2 - t_1 = \int_{t_1}^{t_2} dt = \frac{1}{c}\int\left(\sqrt{\alpha_{ab}\dot{x}^a\dot{x}^b} - \alpha_c\dot{x}^c\right)ds$$

this gives redshift (with $d\tau/dt = e^\phi$)

$$\frac{\nu}{\tilde{\nu}} = \frac{d\tilde{\tau}}{d\tau} = \frac{d\tilde{\tau}}{dt}\frac{dt}{d\tau} = \frac{e^{\tilde{\phi}}}{e^{\phi}}$$

Clock comparison with optical fibers

with refractive index we have a modified “metric”

$$g = e^{2\phi}\left(-\frac{1}{n^2(x)}\left(cdt + \alpha_a(x)dx^a\right)^2 + \alpha_{ab}(x)dx^a dx^b\right)$$

modified redshift

$$\frac{\nu}{\tilde{\nu}} = \frac{e^{\tilde{\phi}}}{e^{\phi}}\frac{n}{\tilde{n}}$$

with known refractive index we can determine the gravitational redshift potential

Relativistic geoid II: Model of the Earth

Earth described by a continuum, relative velocity between constituents can be decomposed

$$v^\mu = \omega^\mu{}_\nu r^\nu + \sigma^\mu{}_\nu r^\nu + \frac{1}{3}\theta r^\mu$$

- a body is called rigid, if all spatial distances and angles between nearby particles remain constant

The rigid body

A non-expanding and shear-free congruence is called rigid.

the rigid body can still rotate and accelerate in a *time-dependent* way

Theorem

If for a rigid body the rotation is constant and the acceleration rotates with the rigid body, then

- the congruence is stationary,
- the acceleration can be derived from a potential.

Relativistic geoid II: Model of the Earth

Definition

A congruence is called stationary if u is proportional to a Killing vector field ξ with $\mathcal{L}_\xi g = 0$. This is also called an isometry.

- $\xi = e^\phi u$
- expansion $\theta = 0$, shear $\sigma = 0$
- $a_\mu = D_u u_\mu = -\partial_\mu \phi$
- potential is time independent: $D_u \phi = 0$

the acceleration of falling bodies (falling corner cubes, plumb lines) is given by a potential $\phi \Rightarrow$

ϕ is a fully general relativistic geoid

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Theorem

Both definitions of a geoid coincide

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Post-Newtonian limit

interpretation

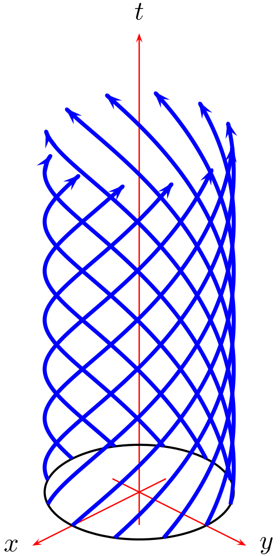
- if w is a geodesic motion and if we define $p = mg(\cdot, w)$, then

$$
\begin{aligned}
const = E = p(\xi) = mw(e^{\phi}u) &= e^{\phi}w(u) \\
&= e^{\phi}\frac{1}{\sqrt{1-\dot{x}^2}} \\
&\approx m + \frac{m}{2}\dot{x}^2 + m\phi + \ldots
\end{aligned}
$$

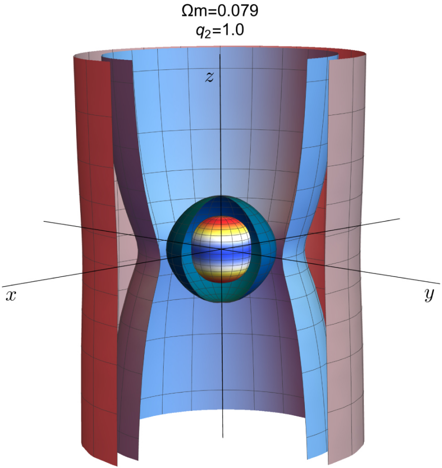
  ϕ plays the role of a gravito-inertial potential

- a is the acceleration of a body at rest (of a body on the surface of the Earth, ma is the weight or “gravito-inertial force” of the body)
- to first order $\phi = W/c^2$ (Newton + rotation)

can be measured by free fall experiments (falling corner cube)



Geoid in Erez-Rosen space-time



Philipp, Hackmann, Perlick, Puetzfeld, C.L., PRD 2017

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General relativistic geoid: open problems

Theoretical problems

- series expansion with respect to which multipoles? Complete set. Test bed: stationary multipoles.
- generalized gravitational potential is one function: Relativistic gravity has 10 quantities: What is the role of the other 9 functions? Are they needed? How can they be measured? Which role do they play for geodesy? e.g.:
  - with the gravitomagnetic field you should see moving masses, the **mass currents**; in the Einsteinian framework we have more information than in Newtonian framework
  - one also should see pressure, stress, ...
- use of moving clocks like clocks in space: how **moving clocks** can measure the geoid?
- and what does **GRACE** measure?
- do we need an ellipsoid?

Practical problems

- local and global leveling
- time dependent contributions

Clock comparison space–ground

General problem: clock and frequency comparison between clock n ground (rotating) and clock on a satellite (orbiting he earth)

- clock on rotating Earth
- satellite moving on geodesic. in Schwarzschild, Kerr: r, φ, ϑ, t, s given Weierstrass elliptic functions $\wp$,σ, ζ
- electromagnetic signal between satellite and Earth moves on geodesic: emitter-receiver problem

simplified model

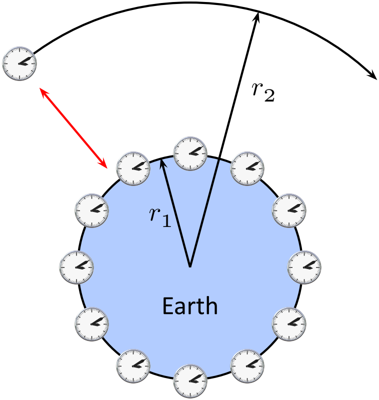
- radial signals
- Schwarzschild orbits

$$r(\varphi) = \frac{2M}{\wp(\varphi) + \frac{1}{6}}$$

Timing: special case

slightly simplified case

- clock orbiting the Earth in free fall
- analytically given orbit
- Schwarzschild geometry
- time comparison through vertical light rays



result

$$\frac{\nu_1}{\nu_2} = \frac{1}{\sqrt{1 + \frac{M}{r_2 - 3M}}} \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}}$$

has to be generalized to arbitrary links for 2nd pN approximation of general case see Linet & Teyassandier, PRD 2002

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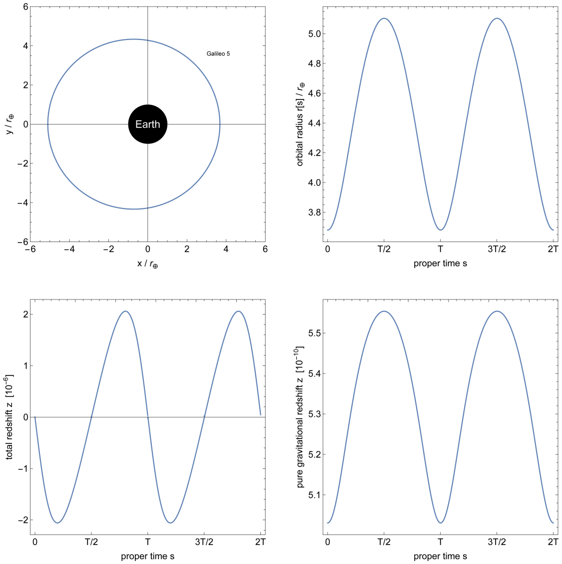
First analytic results



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Summary and main aims

Summary:

- Geometric definition of a “good” clock
- Experimental realization through atomic clocks
- Practical applications
  - metrology (with impact on, e.g., astronomy)
  - positioning
  - geodesy, ...

Principal results and questions:

- develop generally valid notions
- develop a generally valid framework
- outline general applications, effects, measurements
- use that for testing GR

only then one will have a thorough understanding of the field

Outlook - further issue: fundamental physics

Equivalence principle

- Do all atoms fall at the same rate? (Schlippert et al, PRL 2014)

Quantum time

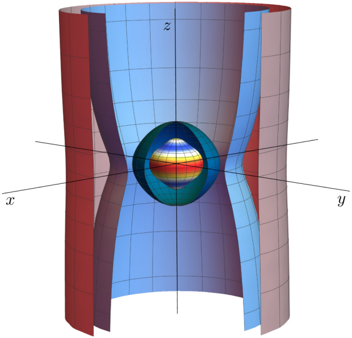
- Is the Compton frequency of an atom a clock? “A rock as a clock” (Müller, Peters, Chu, Nature 2010)

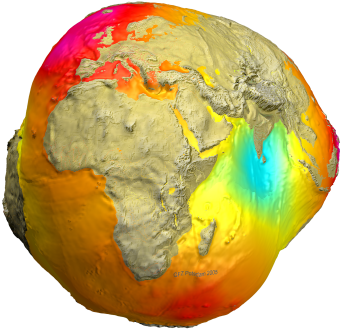
Universality of Gravitational Redshift

- Clock comparison experiments (various experiments in 2010 - ...)

highest precision always requires fundamental physics research: test of SR, GR, search for anomalous interactions, violated symmetries, ...

Thank you!





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