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Simultaneous Determination of Position and Gravity from INS/DGPS

Klaus-Peter Schwarz

Department of Geomatics Engineering, University of Calgary, Canada

1 Introduction

During the past fifteen years, kinematic methods for determining the Earth's surface and its gravity field have progressed from concepts to commercial applications. Currently, the two major areas of application are georeferencing and kinematic gravimetry. In both cases an Inertial Measurement Unit (IMU) and GPS receivers working in differential mode (DGPS) are used as measurement systems. In georeferencing, the output of the two systems is combined to optimize the estimation of the position and orientation of the moving vehicle. In kinematic gravimetry the difference between the output of the IMU and the DGPS is used to optimize the estimation of the gravity disturbance vector.

In general, the two applications are treated as separate tasks, although they use the outputs of the same system combination (IMU/DGPS). There are two reasons for this. One is that the mathematical models used are quite different. In georeferencing the two data streams are optimally combined, while in kinematic gravity they are differenced. The second reason is that the error optimization is quite different in the two applications. In georeferencing the aim is to minimize the errors in the estimation of position and orientation of the vehicle trajectory. In kinematic gravimetry the goal is to minimize the errors in vehicle acceleration and specific force. It makes therefore sense to treat the two problems separately. However, there are situations where it is advantageous, to obtain both the vehicle trajectory and the gravity disturbance vector along its path with highest possible accuracy. In that case, the simultaneous solution of the two problems is of advantage because it allows to consider nonlinearities in the solution. This problem will be discussed in the following.

2 Modeling Rigid-Body Motion

This chapter will serve as an introduction to the concept of modeling rigid-body motion in three-dimensional space. The dynamics considered is that of Earth-bound vehicles, such as airplanes, cars, and ships. Therefore, the Conventional Terrestrial Reference System, denoted by (e), is chosen for mathematical modeling; for a definition, see for instance Hofmann-Wellenhof et al. (1997). The measurement frame is given by the body frame (b), which is defined with respect to the moving vehicle. The transformation from the measurement frame to the Earth-fixed e-frame describes the process of geo-referencing. It requires the determination of the platform motion with respect to the Earth-fixed frame. Motion is described in the usual way by six time-varying parameters. They are chosen as three position parameters (translations) and three orientation parameters (rotations).

The trajectory of the moving object is estimated by kinematic measurements. Thanks to the development of inertial and satellite technology, the measurement of vehicle motion with high accuracy and data rate, has become possible. In the case of particle motion, the trajectory can be expressed by a time-variable position vector. When derived from discrete measurements, a time series of discrete positions will result. It can be approximated by a continuous trajectory of the form

$$\mathbf{r}(t) = \{x(t), y(t), z(t)\} \quad , \tag{1a}$$

if certain smoothness conditions are imposed. Alternatively, a time-variable velocity vector

$$\dot{\mathbf{r}}(t) = \left\{\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\} = \{v_x, v_y, v_z\} \quad , \tag{1b}$$

or a time-variable acceleration vector

$$\ddot{\mathbf{r}}(t) = \left\{\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right\} = \{a_x, a_y, a_z\} = \{\dot{v}_x, \dot{v}_y, \dot{v}_z\} \tag{1c}$$

can also be used if the appropriate initial conditions are specified. In equations (1b) and (1c) a dot above a variable indicates differentiation with respect to time. Each of these equations describes particle motion in an inertial frame of reference. The choice of one of these specific forms obviously depends on the type of observables available and the desired application. Thus, for GPS measurements, the form (1a) is usually the most adequate, while for acceleration determination or kinematic gravimetry the form (1c) is more suitable.

The models discussed up to now describe the trajectory of a particle. In general, the objects whose motion has to be modeled are three-dimensional and of finite extension. They cannot be adequately modeled by particle motion. If the body is rigid, the appropriate model is rigid body motion. It is shown in Figure 1 as the sum of two vectors. One models the position vector from the ground receiver to the aircraft centre of mass in the e-frame and will be denoted by $\mathbf{r}_o^e(t)$. This vector corresponds to the vector previously used for particle motion and describes a translation in three-dimensional space. The second vector $\Delta\mathbf{r}^b$ is defined between the aircraft centre of mass and an arbitrary point on the rigid body. It thus describes a vector fixed in the rigid body and is fixed with respect to the b-frame. It is either called the offset vector or the lever arm. If the b-frame rotates with respect to the e-frame, a rotation matrix $\mathbf{R}_b^e$ is needed to transform the motion of the rigid body back to the e-frame. Thus, the equation for rigid body motion is

$$\mathbf{r}^e(t) = \mathbf{r}_o^e(t) + R_b^e(t)\Delta\mathbf{r}^b \tag{2}$$

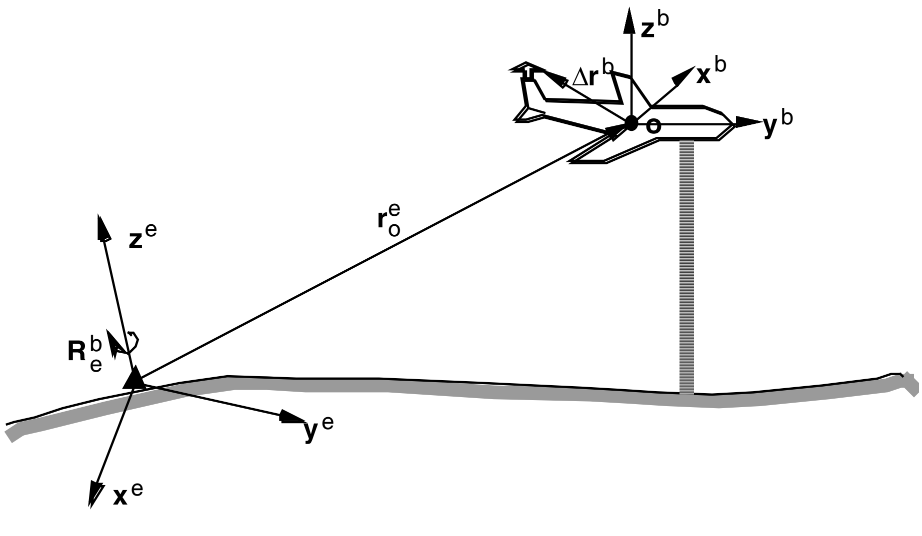

Figure 1: Rigid body motion

Note that both the translation vector $\mathbf{r}_o^e$ and the rotation matrix $\mathbf{R}_b^e$ are time dependent while the vector $\Delta\mathbf{r}^b$ is not. Equation (2), is often called the georeferencing equation. It can be easily shown that formula (2) still applies when none of the two sensor locations is at the centre of mass of the moving object. As long as the rigid connection between the sensors can be assumed as known, the formula is valid. A large class of kinematic modeling problems in geodesy can be defined as the determination of time variable position and orientation of a moving sensor with respect to an Earth-fixed reference system.

3 The INS/GPS Core Solution

The two types of measuring systems that will be considered here are a strapdown IMU and differential GPS. This can be considered as the standard equipment and can serve as the basis for modeling more complex systems. In this setting, the aircraft is considered as a rigid body and the two points on it are the IMU centre and the antenna centre of the airborne GPS receiver. The vector $\Delta\mathbf{r}^b$ is the coordinate difference between the two centres in the b-frame. Using these measurements, the unknowns in equation (2) can be determined.

A strapdown IMU outputs three components of the specific force vector and three components of the angular velocity vector in the body frame. They will be denoted by $\mathbf{f}^b$ and $\boldsymbol{\omega}_{ib}^b$ in the following. The subscripts of the angular velocity vector indicate the direction of the rotation, the superscript the frame in which the vector is expressed. In this case, the b-frame is rotating with respect to the i-frame, and the vector is coordinated in the b-frame. The i-frame is a properly defined inertial reference frame in the Newtonian sense, and, thus, can be considered as being non-accelerating and non-rotating with respect to the distant galaxies. Specific force and angular velocity can be used to formulate a system of differential equations in a rotating frame from which all parameters can be obtained that are required to describe rigid body motion. The system is of the following form:

$$\begin{pmatrix} \dot{\mathbf{r}}^e \\ \dot{\mathbf{v}}^e \\ \dot{\mathbf{R}}_b^e \end{pmatrix} = \begin{pmatrix} \mathbf{v}^e \\ \mathbf{R}_b^e \mathbf{f}^b - 2\boldsymbol{\Omega}_{ie}^e \mathbf{v}^e + \mathbf{g}^e \\ \mathbf{R}_b^e (\boldsymbol{\Omega}_{ib}^b - \boldsymbol{\Omega}_{ie}^b) \end{pmatrix} \tag{3}$$

where (e) is the Conventional Terrestrial Coordinate System. For a derivation of this formula, see Britting (1971) or Schwarz (2000).

The vector on the left-hand side is called the state vector and has nine components, three for position, three for velocity, and three for orientation. It describes the changes of these components with time. To solve the system, the observables $\mathbf{f}^b$ and $\boldsymbol{\omega}_{ib}^b$ are needed as well as, the gravity vector $\mathbf{g}^e$, the Earth rotation rate $\boldsymbol{\omega}_{ie}^b$, and the dimensions of the reference ellipsoid. The specific force vector $\mathbf{f}^b$ is directly used, while the angular velocity vector is contained in $\boldsymbol{\Omega}_{ib}^b$, which is a skew-symmetric matrix of the vector elements. It is used to determine $\mathbf{R}_b^e$ by integration. The gravity vector can be approximated by the so-called normal gravity model, while Earth rotation is known with sufficient accuracy. As can be seen from the second set of equations, rotational and translational parameters are needed to integrate the velocity equations. Since position is obtained by a direct integration of velocity, position and rotation parameters are interrelated. When solving equation (3), modeling errors have to be taken into account. They include accelerometers biases **b** and gyro drifts **d**.

GPS observables are either of the pseudorange type ρ or of the carrier phase type Φ. Models to transform the resulting range equations into positions and velocities are well-known, see for instance Hofmann-Wellenhof et al. (1997). In the process, orbital models as well as atmospheric models are needed and the Earth rotation rate is again assumed to be known. To facilitate comparison with the INS model, the GPS-trajectory equations will not be expressed in terms of the original observables but in terms of position and velocity which can be considered as GPS pseudo-observables. The trajectory model is then of the form

$$\begin{pmatrix} \dot{\mathbf{r}}^{e} \\ \dot{\mathbf{v}}^{e} \end{pmatrix} = \begin{pmatrix} \mathbf{v}^{e} \\ \mathbf{0} \end{pmatrix} \tag{4}$$

for differential pseudorange or carrier phase measurements.

In principle, the first set of equations (4) is sufficient to model the translation vector $\mathbf{r}^e$ from carrier phase observables if constant velocity between updates can be assumed. However, experiments have shown that the observation of phase rates improves the estimation and that therefore the full model (4) is more appropriate. It assumes constant acceleration between measurement epochs. This assumption is certainly justified for the high internal data rates of GPS receivers. In cases where the output data rate is used for trajectory interpolation, other models may be more appropriate, see for instance Schwarz et al. (1989).

The formulation of a state vector model has the advantage that standard methods, such as Kalman filtering, are available to estimate $\mathbf{r}^e$ and $\mathbf{R}_b^e$ directly as functions of time. This is straightforward when using IMU measurements for the determination of position, velocity, and orientation. When using GPS measurements, position and velocity estimates are obtained at each measurement epoch. Thus, the trajectory could be determined by fitting a curve through a string of estimated positions and associated velocity vectors from the GPS observations. If the statistics are properly taken into account, this will result in a perfectly acceptable solution. However, IMU and DGPS measurements can also be combined by using the GPS measurements as updates to the state vector (3). By defining covariances and spectral densities for the parameters, smoothness conditions are automatically imposed on the solution. From a conceptual point of view, this model offers a convenient way to discuss and implement GPS/INS integration.

GPS and INS are in many ways complementary systems for accurate position and orientation determination. GPS positioning by differential carrier phase is superior in accuracy as long as no loss of lock occurs. GPS relative positions are therefore ideally suited as updates to equation (3), in order to prevent systematic error growth in the INS trajectory. On the other hand, IMU measurements are very accurate in the short term and can thus be used to detect and eliminate cycle slips or bridge short-term loss of lock. Because of its high data rate, an IMU provides a much smoother interpolation. It also gives the rotation matrix $\mathbf{R}_b^e$ with higher accuracy than DGPS multi-antenna methods. Integration via a Kalman filter seems thus to be appropriate, using IMU data for the basic integration and GPS for the updating.

4 Georeferencing and Kinematic Gravimetry

As can be seen from equation (2) and the associated Figure 1, the origin of the b-frame is located in the centre of the IMU, which is usually taken as the origin of the orthogonal accelerometer triad. The gyro triad, sensing the angular velocities, is aligned to it. Since the

IMU is fix-mounted to the rigid aircraft body, all inertial measurements are given in the b-frame. The DGPS measurements refer to the origin of the antenna centre of the airborne GPS receiver and are given in the e-frame. The vector pointing from the INS centre to the antenna centre of the airborne receiver is denoted by $\mathbf{a}^{GPS}$. Thus, the position of the IMU centre can be obtained from

$$\mathbf{r}^{e}_{INS}(t_j) = \mathbf{r}^{e}_{GPS}(t_j) - \mathbf{R}^{e}_{b}(t_j)\mathbf{a}^{GPS} \tag{5}$$

where $\mathbf{r}^{e}_{GPS}(t_j)$ is the position of the GPS antenna center in the m-frame and $\mathbf{R}^{e}_{b}(t_j)$ is the rotation matrix between the b-frame and the m-frame measured at time (t_j). This matrix is directly determined during INS data processing.

Equation (5) is the simplest form of the georeferencing equation. It describes the trajectory of the IMU centre in the Conventional Terrestrial Frame and the orientation changes of the inertial sensors. In many cases, an imaging sensor is added to the measuring system, in order to determine the topographic surface of the Earth from airborne kinematic measurements. For a detailed description of this application, see Schwarz et al. (1993), Mostafa and Schwarz (2000) and El-Sheimy (1996).

For kinematic gravimetry, the model equations are of type (1c), i.e. they have to be formulated in the acceleration domain, not the position domain. Thus, the time-variable position vector r^e has to be differenced twice to obtain aircraft acceleration. As a result, the effect of measurement errors in positioning and gravity determination is very different. For instance, DGPS errors, that are negligible for positioning, become critical for gravity determination because they are amplified by the double differentiation. Similarly, white noise in the acceleration measurements will directly show up in the gravity estimate while it is usually negligible for positioning because of the smoothing effect of double integration. The overall effect is that gravity estimates contain high-frequency noise with large amplitudes which limit the resolution of the high frequency spectrum of the gravity field.

The gravity model can be derived by taking the second row of equation (3) and rewriting it with respect to gravity

$$\mathbf{g}^{e} = \dot{\mathbf{v}}^{e} - \mathbf{R}^{e}_{b}\mathbf{f}^{b} + 2\Omega^{e}_{ie}\mathbf{v}^{e} \tag{6a}$$

The total gravity vector can be split into a normal gravity component γ and a gravity disturbance $\delta\mathbf{g}^{e}$. The normal component represents the gravity field of an ellipsoid of revolution which has the same total mass as the Earth and rotates with the mean angular velocity as the Earth. It is easy to compute normal gravity by compact formulas. The gravity disturbance is the difference between actual gravity and normal gravity computed at the same point

$$\delta\mathbf{g}^{e} = \mathbf{g}^{e} - \gamma^{e} \tag{6b}$$

Normal gravity can easily be obtained from a model. It is thus convenient to consider the gravity disturbance as the unknown. Equation (6a) can therefore be written

$$\delta\mathbf{g}^{e} = \dot{\mathbf{v}}^{e} - \mathbf{R}^{e}_{b}\mathbf{f}^{b} + 2\Omega^{e}_{ie} \bullet \mathbf{v}^{e} - \gamma^{e} \tag{6c}$$

An explanation of the terms in this equation is given in the context of equation (3). If the complete vector $\delta\mathbf{g}^{e}$ is determined, we speak of vector gravimetry. If only its magnitude is of interest, we speak of scalar gravimetry. For different concepts of scalar gravimetry and the corresponding equations, see Schwarz and Wei (1990) and Wei and Schwarz (1997).

An accelerometer does not measure gravity directly, but measures the total effect generated by all contact forces working on it. To obtain gravity, the effects of vehicle acceleration and frame rotation have to be subtracted. Vehicle acceleration in the e-frame is given by the $\dot{\mathbf{v}}^e$-term. Frame rotation results in the Coriolis term $2\Omega^e_{ie}\mathbf{v}^e$. Thus, by rotating the measurement from the b-frame to the e-frame and subtracting the two acceleration terms, as well as normal gravity, the gravity disturbance is obtained. This means that besides position and orientation which are required in georeferencing, velocity and acceleration are needed as additional parameters in kinematic gravimetry. It should be noted that vehicle position, velocity, and acceleration must all be determined from DGPS, because the accelerometer outputs are used for the determination of specific force.

The gravity disturbance model is often formulated in the local-level frame where it has the form

$$\delta\mathbf{g}^l = \mathbf{v}^l - \mathbf{R}^l_b\mathbf{f}^b + (2\Omega^l_{ie} + \Omega^l_{el}) \bullet \mathbf{v}^l - \gamma^l \tag{7}$$

The additional velocity-dependent term in this equation is due to the additional rotation effect generated by the use of a curvilinear coordinate system. It can be visualized by considering an airplane flying at constant height above the ellipsoid. The additional angular velocity Ω^l_{el} is dependent on the curvature of the ellipsoid and the aircraft speed.

As in the case of georeferencing the offset between the IMU centre and the airborne DGPS antenna centre has to be taken into account. In this case the effect of the offset on specific force measurements has to be considered. It is of the form

$$\Delta\dot{\mathrm{v}}^l = (\dot{\Omega}^l_{be} - \Omega^l_{be}\Omega^l_{bl})\mathrm{R}^l_b \bullet \Delta\mathrm{r}^b \tag{8}$$

where $\Delta\dot{\mathrm{v}}^l$ are corrections to the DGPS-derived accelerations and $\mathbf{\Delta r}^b$ is the offset vector (lever arm) expressed in b-frame coordinates. A derivation can be found in Knickmeyer (1990). The effect on the gyro measurements has been neglected, because the vehicle body is considered as rigid. In that case, the rotations measured at one point on the body are the same on all other points.

5 Iterating the Integrated Solution

Equations (5) and (6) describe georeferencing and kinematic gravimetry, respectively. In order to obtain the best possible estimates of all three vectors (position, orientation, and gravity disturbance), the two sets of equations have to be combined. There are a number of different models that can be used for this purpose. For simplicity, the iterative solution shown in Figure 2 will be presented.

In this Figure the IMU output is shown on the left-hand side and the DGPS output on the right-hand side. At the measurement level the IMU observables $\mathbf{f}^b$ and $\boldsymbol{\omega}^b_{ib}$ are shown, as well as the initial values required for the integration process ($\mathbf{R_0}$, $\mathbf{v_0}$, $\mathbf{r_0}$, γ) Similarly, in the DGPS stream, the observables pseudo-range, range rate, and phase are shown, together with the orbital and atmospheric models needed for the position determination. At the integration level, the alignment values for the IMU are obtained and are used to get the transformation matrix $\mathbf{R}^e_b(t_j)$ by integration, using the observables and the normal gravity field to determine position $\mathbf{r}^e$ and velocity $\boldsymbol{\omega}^e$ of the aircraft. Similarly, position, velocity, and acceleration are obtained from the DGPS data stream. An appropriate model for the estimation of acceleration from the position and velocity data has to be formulated.

At this point the two data streams are brought together and are differenced, in order to use the position and velocity differences as updates for bias estimation ($\hat{\mathbf{b}}^e$ **,** $\hat{\mathbf{d}}^e$) in the Kalman filter. The estimated biases are used to correct the IMU observables and to obtain $\bar{\mathbf{f}}^e$ and $\bar{\boldsymbol{\omega}}^e$. The acceleration $\dot{\mathbf{v}}^e$ computed as part of the DGPS data stream is then subtracted from the corrected specific force measurement $\bar{\mathbf{f}}^e$ to obtain $\delta\mathbf{g}^e$. The most recent estimate of $\delta\mathbf{g}^e$ is compared to the previous one. If the change is smaller than a pre-determined threshold value, the current values for the trajectory parameters (position and orientation), the gravity disturbance vector, and the bias terms are considered as final.

If the change is larger than the threshold value, a new iteration is started, in which the currently best estimates for the trajectory parameters, the gravity vector (γ^e + $\delta\mathbf{g}^e$), and the bias terms are used. Note that the DGPS parameters have not to be re-computed. Only the estimates that are affected by the IMU data stream will change.

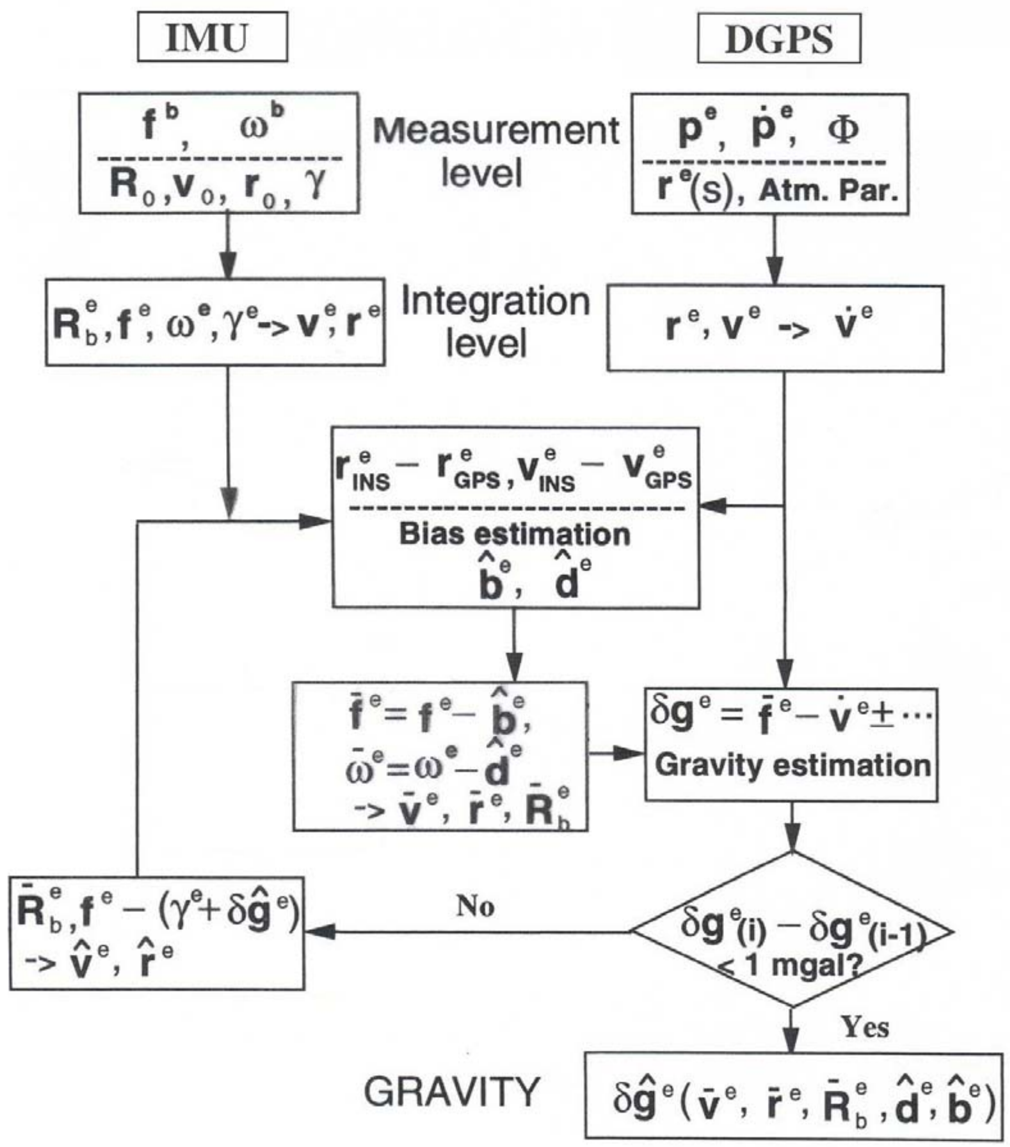

Figure 2: Iterating the integrated solution

The approach presented in Figure 2 is essentially a refinement of the method usually applied in airborne gravimetry, see for instance Bruton (2000). Instead of using only the normal gravity model, an iteration process is set up to approximate actual gravity along the flight trajectory. As $\mathbf{g}^e$ changes, all estimates in the IMU data stream also change. Due to the weak nonlinearity, convergence will be rather rapid. Both, scalar and vector gravimetry will profit from this refinement. As global gravity models from dedicated gravity satellites become available, airborne methods will gain in importance. They will increase the resolution of the high-frequency spectrum of the gravity field which is not covered by current gravity satellites. Augmenting the measuring system by an imaging sensor would allow the recovery of the topography under the flight area with high accuracy. This would be useful in geoid modeling and in geophysical prospecting, see Schwarz and Li (1996). Currently, no use is made of the horizontal components of the gravity disturbance vector, because they cannot be estimated with the same accuracy as the vertical component, see for instance Schwarz et al. (2001) for some results. Once the estimation of the horizontal components can be improved to the same level, a direct method for geoid profiling from the air would be available. This would make local geoid determination more versatile and efficient. In summary, the capability of simultaneously determining the surface of the Earth and its gravity field could be used with advantage in a number of geodetic applications.

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