Stokes Integration versus Wavelet Techniques for Regional Geoid Modelling

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Abstract. For the computation of high resolution regional geoid models, gravity and terrain data in connection with a global geopotential model play a very important role. The data sets are usually combined in a remove-restore procedure. In many cases, the transformation from gravity anomalies to geoid undulations is done using Stokes's integration kernel or a modified integration kernel, e.g., based on the spectral combination technique. Least squares collocation may be used for this task as well, but for continental-scale computations the integration techniques are often preferred due to their high computational efficiency.

Besides the classical integration techniques, the wavelet technique is investigated in this contribution. The wavelet technique also uses residual gravity field quantities in a remove-restore procedure. However, the computations are carried out in two steps. The first step consists of a convolution of the residual gravity data with several wavelet functions, being contracted or dilated variants of one prototype ("mother") wavelet function. This leads to a decomposition of the whole spectrum of the original data into a set of filtered detail signals with unique spatial resolution. This type of space and frequency analysis is called multi-scale analysis (MSA). The second step then convolves the residual gravity details with an integration kernel (e.g., Stokes) and leads to corresponding geoid undulations. The second step, applied to every decomposed detail (scale) of the original data, corresponds to the classical integration techniques.

In this contribution, both the classical integration and spherical wavelet techniques are applied using Europe as a test area. The differences in methodology and numerical performance of both techniques are investigated. Finally, the results are evaluated by independent GPS and levelling control points.

Keywords. Stokes, multi-scale analysis, spherical wavelets

1 Introduction

The determination of high resolution regional geoid models is accomplished by the combination of various gravity field observations, all of them having unique properties in terms of data coverage and spectral behavior. Usually a global gravity field model, terrestrial gravity data and terrain data are combined in a remove-restore procedure, where the long wavelength content of the global model and the short wavelength content of a topographic model are subtracted from the surface gravity data. The resulting residual gravity anomalies contain mainly the medium wavelengths of the gravity spectrum and have small values.

The conversion of these residual gravity anomalies to geoid undulations can be realized by different techniques. One method is least squares collocation, but for continental-scale computations a second method, the integration technique, is preferred due to its higher computational efficiency. The integration is done using Stokes's integration kernel or a modified integration kernel, e.g., based on the spectral combination technique. A third technique for geoid computation is the wavelet modelling, see Freeden and Schneider (1998). It is also based on Stokes's formula and therefore partly comparable to the integration technique.

In this paper we will discuss and compare the methodology of both the integration and the spherical wavelet technique. Results are presented for both methods using a $5' \times 5'$ gravity anomaly grid from the European Geoid Project, Denker and Torge (1998). Finally the pros and cons of both methods are outlined and a summary of the comparisons is given.

2 Methodology

2.1 Stokes Integration

The classical way to transform gravity data collected at the surface of the Earth into geoid undulations N was found by Stokes (1849). His formula performs

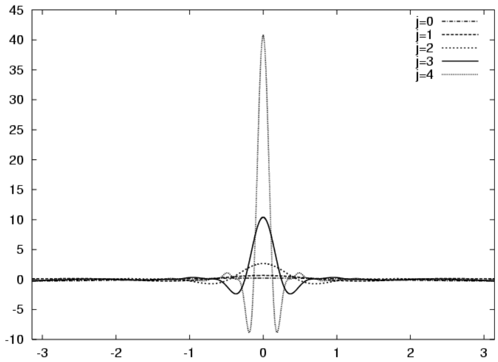

Fig. 1. Smoothed Shannon wavelet function $\Psi_j(\psi)$ for scales $j = 0 \ldots 4$

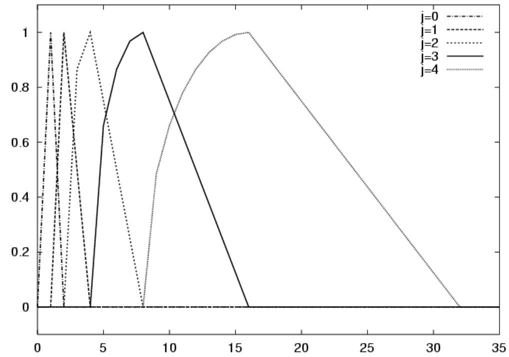

Fig. 2. Legendre coefficients $\Psi_j^\wedge(\ell)$ of smoothed Shannon wavelet for scales $j = 0 \ldots 4$

the integration of gravity anomalies Δg distributed over the whole sphere using the Stokes function $S(\psi)$ as integration kernel, which acts as a weighting function for the anomalies.

$$N = \frac{R}{4\pi\gamma} \int\limits_{\Omega} S(\psi)\, \Delta g\, d\omega \tag{1}$$

In (1) ψ is the spherical distance between the computation point and the surface element $d\omega$ with the gravity anomaly Δg. Furthermore, γ is the mean normal gravity value over the Earth of radius R. The integration over the whole sphere Ω can be regarded as a spherical convolution of the function Δg with the kernel S, see Freeden et al. (1998). Using this relation equation (1) becomes

$$N = \frac{R}{4\pi\gamma} (S * \Delta g). \tag{2}$$

This convolution can be computed fast and exactly via a 1D Fourier transform, see Haagmans et al. (1993). The Stokes function $S(\psi)$ depends only on the spherical distance ψ and can be computed via a closed formula. The drawback of this function is its infiniteness for $\psi = 0$, so that a special treatment of the neighborhood of the computation point is necessary.

When using residual data, the computation over the whole sphere can be limited to a bounded region under the assumption of zero-values outside this region, e.g. Torge (2001).

2.2 Wavelet Modelling

Wavelets are applied in a wide range of different branches. The common basis for all of them is the wavelet function, which is defined via the contraction and translation of one prototype "mother" wavelet. Various wavelet techniques were developed, e.g., through different realizations of the contraction, or the domain of the function. For instance, there exist discrete and continuous wavelet transforms as well as 1- or 2-dimensional or spherical wavelets, see Grabs (1995); Jawerth and Sweldens (1994); Daubechies (1992); Liu and Sideris (2003); Windheuser (1995). In the spherical case, the translation of the "mother" wavelet is replaced by a rotation.

The modelling of the geoid from gravimetric data by wavelets can be achieved by a multi-scale analysis (MSA), see Freeden et al. (1998). Here the contraction is done in fixed steps resulting in discrete scales j. Furthermore, spherical wavelets, depending only on the spherical distance ψ, are used because the spherical shape of the Earth has to be considered for continental scale computations. The spherical wavelet functions are defined by the Legendre series

$$\Psi_j(\psi) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \Psi_j^\wedge(\ell) P_\ell(\cos\psi), \tag{3}$$

where the Legendre coefficients $\Psi_j^\wedge(\ell)$ are connected to the generating "mother" wavelet. An example of a wavelet function and its corresponding Legendre coefficients are illustrated in Figs. 1 and 2.

The wavelet modelling by MSA consists of two steps. The first step is the decomposition of the source data, or generally a function F, into its wavelet transforms of scales j

$$\mathrm{WT}\{F\}_j = (\Psi_j * F) = \int\limits_{\Omega} \Psi_j\, F\, d\omega. \tag{4}$$

This step can be interpreted as a bandpass-filter. The function F is split up into several spectral bands. The bandwidth of each scale j is given by the range of degrees ℓ where the Legendre coefficients $\Psi_j^\wedge(\ell)$ are unequal 0, see Fig. 2.

The second step of the MSA is the reconstruction of the wavelet transform $\mathrm{WT}\{F\}_j$ to the detail signal F_j of scale j. The summation of all detail signals yields the original function F.

$$
\begin{aligned}
F &= \sum_{j=0}^{\infty} \left(\mathrm{WT}\{F\}_j * \tilde{\Psi}_j \right) \\
&= \sum_{j=0}^{\infty} \int_{\Omega} \mathrm{WT}\{F\}_j \; \tilde{\Psi}_j \, d\omega = \sum_{j=0}^{\infty} F_j
\end{aligned} \tag{5}
$$

In the reconstruction, the dual wavelet function $\tilde{\Psi}$ is used, which relates to Ψ via the "refinement equation". In case of the P-scale discrete wavelets used here, Ψ and $\tilde{\Psi}$ are identical, for further details see Freeden et al. (1998).

With the help of the equations (4) and (5) one can decompose a function F into its wavelet transform and exactly reconstruct the same function F from this transform. If we insert the gravity anomalies Δg for function F and put (5) in (1), we get

$$
\begin{aligned}
N &= \frac{R}{4\pi\gamma} \sum_{j=0}^{\infty} \left(\mathrm{WT}\{\Delta g\}_j * \left(S * \tilde{\Psi} \right)_j \right) \\
&= \frac{R}{4\pi\gamma} \sum_{j=0}^{\infty} \int_{\Omega} \mathrm{WT}\{\Delta g\}_j \left(S * \tilde{\Psi} \right)_j \, d\omega,
\end{aligned} \tag{6}
$$

a reconstruction/transformation formula for the geoid, see Schmidt (2001) and Schmidt et al. (2002). The details of the geoid undulation are determined by a convolution of the wavelet transform of the gravity anomalies with an integral kernel, which itself results from a convolution of the Stokes's function with the wavelet functions of scale j. The integral kernel is illustrated in Fig. 3. Note that it is finite for small distances ψ, thus no inner-zone correction is necessary.

Both steps of the MSA are realized by a spherical convolution. As shown in Sect. 2.1 and equations (4)–(6), the convolution is equivalent to an integration over the data region and can be computed exactly by a 1D Fourier transform. The necessary kernels Ψ_j and $(S * \tilde{\Psi})_j$ have to be computed via an infinite Legendre series, see (3). For band-limited wavelet functions, where the Legendre coefficients are unequal 0 only for a bounded range of degrees ℓ, the summation can be limited to $2^{j+1} - 1$. But, nevertheless, the exact computation for every ψ is very time-consuming, thus a tabulation of the kernels is useful.

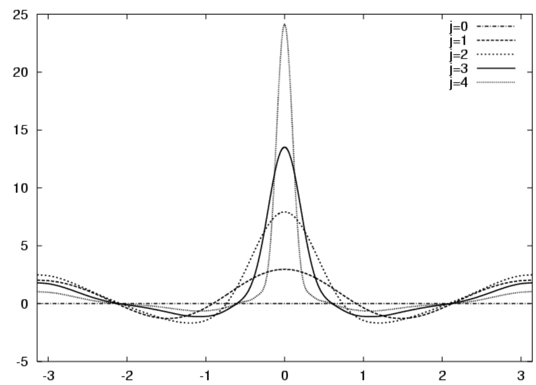

Fig. 3. Integral kernel $(S * \tilde{\Psi})_j(\psi)$ for smoothed Shannon wavelet and scales $j = 0 \ldots 4$

Another problem arises from the infinite summation in the reconstruction formula (6). The total signal is recovered by the summation of *all* detail signals. As the spectral content of the data is limited, the detail signal will vanish for some j, so a truncation is possible. For practical computations the maximum scale is given by the sampling theorem. If the bandwidth of the wavelet function is smaller than twice the resolution of the data, aliasing effects will occur. Consequently, the truncation scale $j_{\max}$ is given by the chosen wavelet function and the data resolution.

2.3 Comparison of Both Methods

Both methods, Stokes integration and wavelet modelling, are based on Stokes's formula. The major difference in methodology consists of the different handling in the frequency domain, leading to a different computational effort.

More precisely, the integration technique handles the whole bandwidth of the residual gravity anomalies at once, so only one convolution is necessary. The wavelet technique first splits the data into several spectral bands by a bandpass-filter and then determines the geoid by handling each spectral band separately. Therefore two convolutions per spectral band are necessary, whereby the second convolution is comparable to the Stokes integration.

3 Results

For the application of both methods, $5' \times 5'$ terrestrial gravity anomalies from the European Geoid Project, see Denker and Torge (1998), were used. The long wavelength part of the gravity field was subtracted using the EGM96 model up to degree and order 360. Furthermore, we subtracted the influence of the topography, determined by a digital terrain model, leading to residual gravity anomalies. The

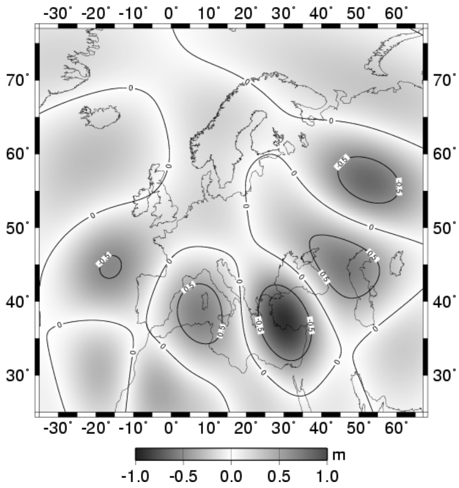

Fig. 4. Detail signal N_4

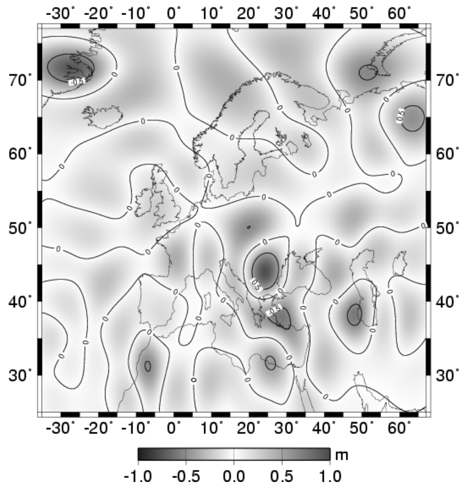

Fig. 5. Detail signal N_5

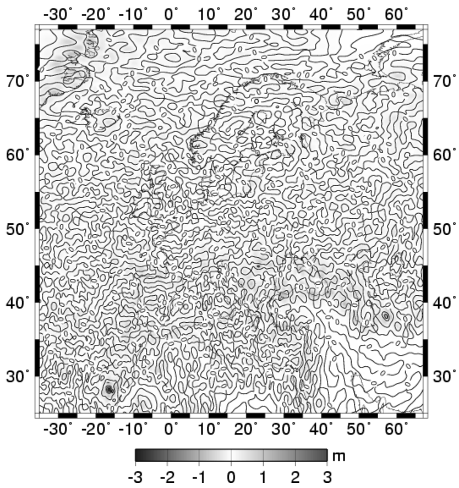

Fig. 6. Detail signal N_8

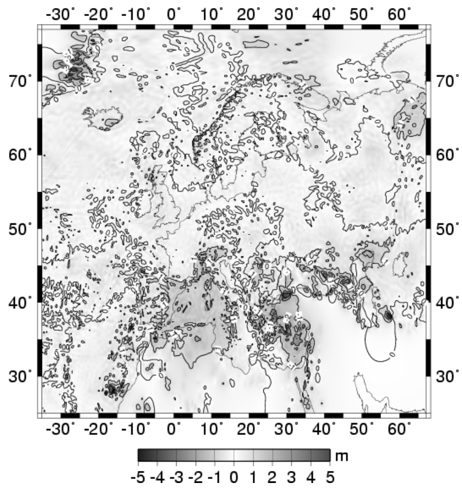

Fig. 7. Sum of detail signals N_j of scales $j = 1 \dots 11$

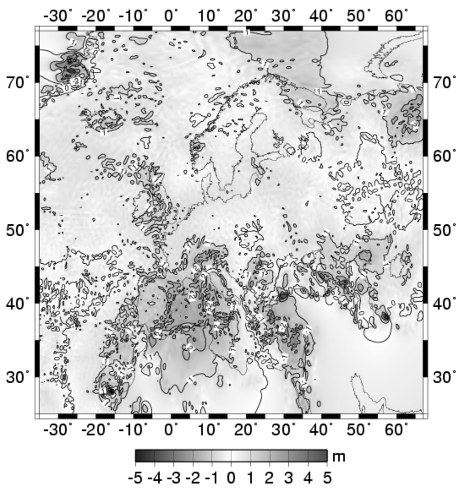

Fig. 8. Geoid undulations determined by Stokes integration

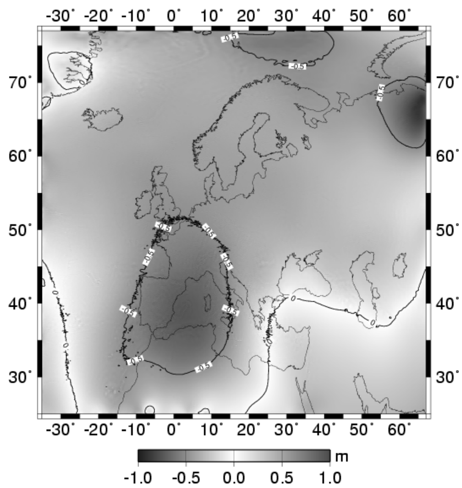

Fig. 9. Difference between Wavelet and Stokes geoid

computation area can be seen in Figs. 4–9.

All results presented here are based on the smoothed Shannon wavelet function with a smoothing factor of $h = 0.5$, illustrated in Fig. 1. The MSA was performed up to a maximum scale of $j = 11$.

In Figs. 4–6 the detail signals, determined by the wavelet technique, are shown for scales 4, 5 and 8 as examples. The figures show a decrease of the wavelengths with increasing scale and a varying amplitude between the scales, which can be attributed to the different energy in the spectral bands.

The sum of all detail signals from scale 1 to 11 is depicted in Fig. 7. This is the final result of the wavelet modelling. The sum contains all degrees up to ℓ=2048. A summation up to scale 12 would contain degrees up to ℓ=4096, but the 5$'$ data resolution only allows a computation up to ℓ=2160, otherwise aliasing effects will occur. This leads to an approximation error, i.e, a loss of spectral content, because the short wavelengths (ℓ=2049–2160) of the residual gravity anomalies are missing. The differences between the wavelet result in Fig. 7 and the solution by classical Stokes integration in Fig. 8 are illustrated in Fig. 9. The differences show a long wavelength characteristics and have a RMS of 37.1 cm. A possible cause of this effect may be a long wavelength effect in the anomalies reconstructed by the wavelets. This is indicated by the geoid undulations computed from the gravity anomaly approximation error, see Fig. 10, which have a similar long wavelength pattern and a similar RMS of 36.6 cm.

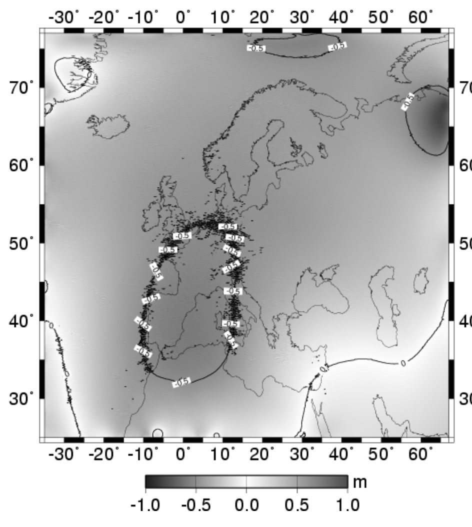

Fig. 10. Geoid undulations from gravity anomaly approximation error

The evaluation of the final results by 166 independent GPS and levelling control points of the UELN (United European Levelling Net), see Ihde et al. (2000), leads to similar results for the Stokes and wavelet method. The bias corrected RMS of the differences between GPS/levelling and the computed geoid undulations are listed in Table 1. The RMS of the wavelet technique is slightly better than that of the Stokes integration.

Table 1. RMS of differences between GPS/levelling and the computed geoid undulations. Units are cm.

Method	RMS
Stokes Integration	49.2
Wavelet Modelling	41.0

4 Summary

In this paper we have shown the similarities but also the differences in methodology of Stokes integration and wavelet modelling in transferring gravity data to geoid undulations.

The main similarity is that both techniques are based on Stokes's formula. From the differences between both methods arise the pros and cons of each technique. The first difference is the computational effort. The wavelet technique takes approx. $2j$ times more time than Stokes. The efficiency may be improved by reducing the integration limits, adapting the localization property of the wavelet function used. The second difference is related to the properties of the kernels. Stokes's kernel is defined via a closed formula, but is infinite at $\psi = 0$, requiring an inner-zone correction. The wavelet kernels are defined by a series expansion. The use of band-limited wavelets and accurate tabulating is necessary to ensure exact and efficient computations. Furthermore, one gains advantage of the finiteness of the wavelet kernels at $\psi = 0$, making further inner zone corrections unnecessary.

A problematic aspect of the wavelet method is its truncation error, arising from the finite summation of the details. This can be reduced by a balanced relation between data resolution and spectral content of the residuals. When raising the data resolution the aliasing effects will be postponed to higher scales, where almost no signal is contained in the residuals due to the terrain reduction. Thus the truncation error will decrease.

The most important advantage of MSA is its great analysis potential, which is, e.g., very useful for the determination of ideal combination solutions from global gravity field models and terrestrial gravity data.

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